SPANNING TREE RESULTS FOR GRAPHS AND MULTIGRAPHS

A Matrix-Theoretic Approach

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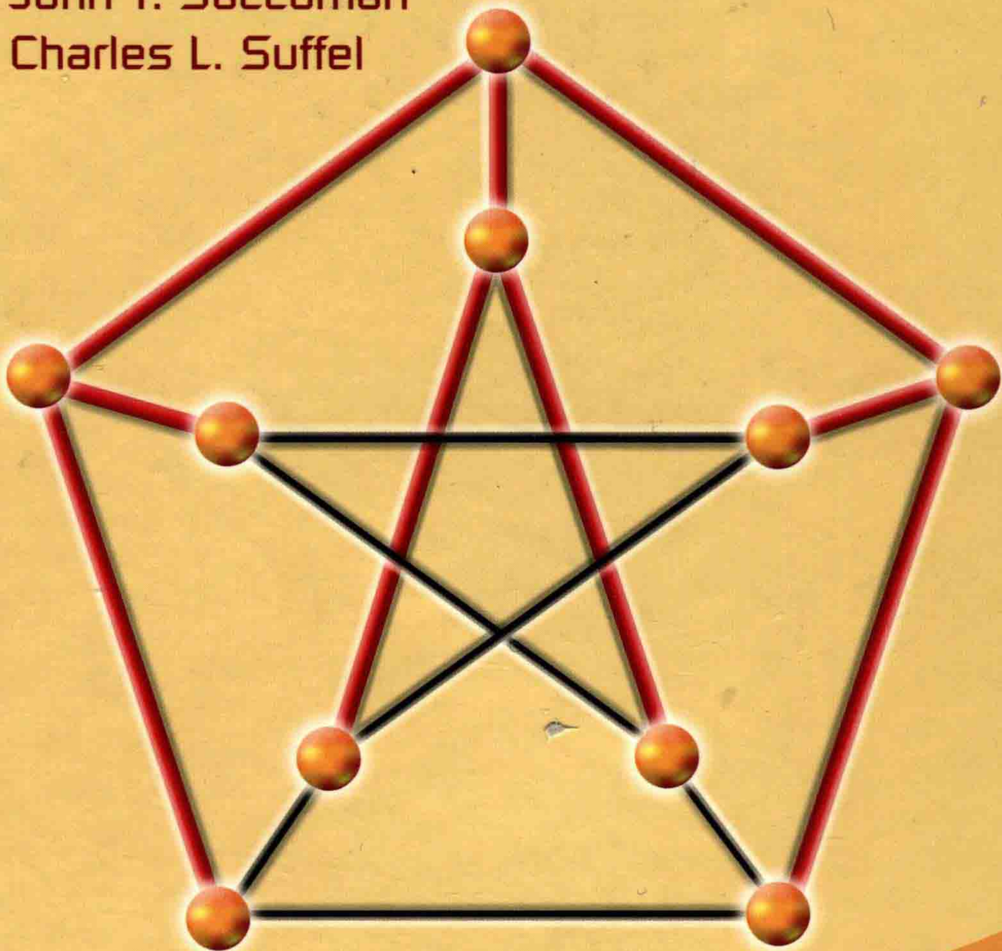

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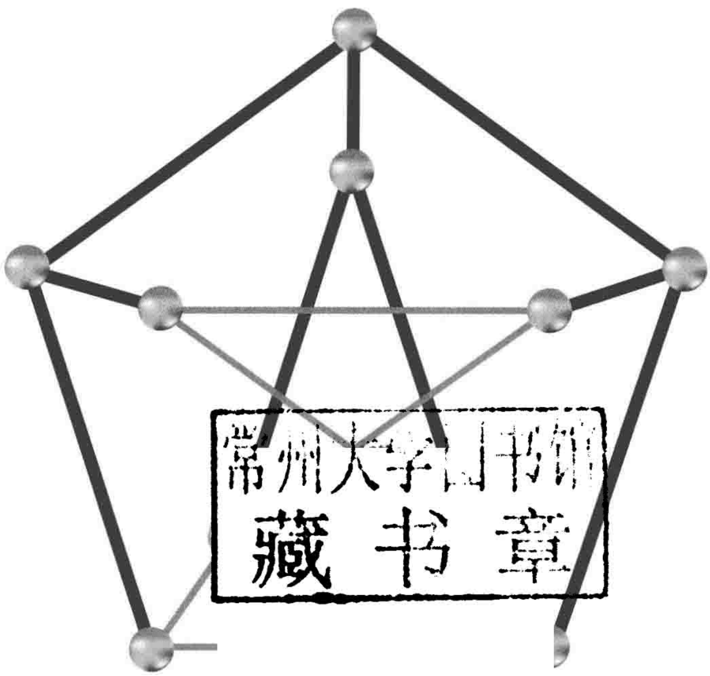

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Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

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ISBN 978-981-4566-03-2

Printed in Singapore by B & Jo Enterprise Pte Ltd

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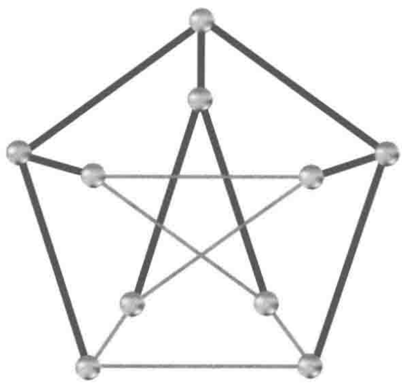

This work could not have been undertaken without the influence of our departed friend and colleague, Frank T. Boesch.

To Mary, the smartest woman I know
--DJG

To John J. Saccoman, my father
--JTS

To my mother Mildred and my father Charles
--CLS

Preface

This book is concerned with the calculation of the number of spanning trees of a multigraph using algebraic and analytic techniques. We also include several results on optimizing the number of spanning trees among all multigraphs in a class, i.e., those having a specified number of nodes, n, and edges, e, denoted $\Omega(n,e)$. The problem has some practical use in network reliability theory. Some of the material in this book has appeared elsewhere in individual publications and has been collected here for the purpose of exposition. A formal probabilistic reliability model can be described as follows: the edges of a multigraph are assumed to have equal and independent probabilities of operation p, and the reliability R of a multigraph is defined to be the probability that a spanning connected subgraph operates. If ς_i denotes the number of spanning connected subgraphs having i edges, then it is easily verified that

$$R = \sum_{i=n-1}^{e} \varsigma_i p^i (1-p)^{e-i}.$$

For small values of p, the reliability polynomial is dominated by the ς_{n-1} term, and since ς_{n-1} is the number of spanning trees, graphs with a larger number of spanning trees will have greater reliability for such p. In the study of graph theory, most of the results regarding the number of spanning trees have only been proven for simple graphs, so herein, we investigate the problem for the extended class of multigraphs. It should be noted that, while extensions to multigraphs make the optimal solution readily apparent in many problems, it is not the case for the spanning tree problem.

In Chapter 0, we present some graph theory and matrix theory background material so that the reader will be familiar with the terminology used in the sequel. In Chapter 1, we introduce many algebraic results for both simple graphs and multigraphs regarding the calculation of their number of spanning trees. In Chapter 2, we present and extend a classical optimization formulation of Cheng that was useful in optimizing the number of spanning trees for certain graphs. In Chapter 3, we present a heretofore unpublished result outlined by the late Frank Boesch in the area of spanning tree enumeration of threshold graphs. In Chapter 4, we show that a complete graph minus a matching, previously shown to have the greatest number of spanning trees among all simple graphs having the same number of nodes and edges, is also optimal when the class is extended to include most multigraphs having a single multiple edge of multiplicity two. We also present an argument using degree sequences that demonstrates the optimality of this simple graph for almost all simple graphs in the class. In Chapter 5, we discuss graphs and multigraphs having all of their Laplacian eigenvalues as integers.

Contents

Chapter 0

An Introduction to Relevant Graph Theory and Matrix Theory

This book is concerned with the calculation of the number of spanning trees of a multigraph using algebraic and analytic techniques. We also include several results on optimizing the number of spanning trees among all multigraphs in a class, i.e., those having a specified number of nodes, n, and edges, e, denoted $\Omega(n,e)$. The problem has some practical use in network reliability theory. Some of the material in this book has appeared elsewhere in individual publications and has been collected here for the purpose of exposition. In preparation, we first collect some relevant graph theoretical and matrix theoretical results.

0.1 Graph Theory

Though we assume that the reader of this work is well versed in Graph Theory, in this section we provide the primary graph theoretic definitions and operations that are used in the body of the succeeding chapters. For any other terminology and notation not provided here we refer the reader to Chartrand, Lesniak and Zhang [Chartrand, 2011].

A *multigraph* is a pair $M = (V, m)$, where m is a nonnegative integer-valued function defined on the collection of all two-element subsets of V, denoted $V_{(2)}$. In the case there are several multigraphs under consideration we use the notation $M=(V(M),m_M)$. The elements of V are called the nodes of the multigraph. We assume that V is a finite set and $n=|V|$ is the *order* of the multigraph, i.e. the order is the number of nodes. A *multiedge* is an element $\{u,v\}\in V_{(2)}$ such that $m(\{u,v\})\neq 0$; $m(\{u,v\})$ is called the *multiplicity* of the multiedge. If for each $\{u,v\}$, $m(\{u,v\})\in\{0,1\}$, then M is a *graph*, and $m^{-1}(\{1\})=E$ is called the *edge set*. In this case, we use G instead of M and employ the alternate notation $G=(V,E)$.

Remark 0.1 The set of multigraphs includes the set of graphs. Thus any result that is stated for multigraphs also holds for graphs. On the other hand, results that are stated for graphs are only applicable to graphs and do not hold for multigraphs.

A multigraph has a geometric representation in which each element (node) of V is depicted by a point, and two points u and v are joined by $m(\{u,v\})$ curves. Figure 0.1 shows the geometric representation of a multigraph M and a graph G. It is traditional to refer to each point in the representation as a node and the collection of all such points as V. Also we refer to each curve in the representation as an *edge*, and the collection of all such curves as E. In the case that $m(\{u,v\})\geq 1$ we represent any single edge between the nodes u and v by uv. The number of edges, denoted by e, is the *size* of the multigraph, i.e. $e=|E|=\sum_{\{u,v\}\in V_{(2)}} m(\{u,v\})$.

We denote the *class of all multigraphs of order n and size e* by $\Omega(n,e)$. In Figure 0.1 $M \in \Omega(5,10)$ and $G \in \Omega(5,7)$

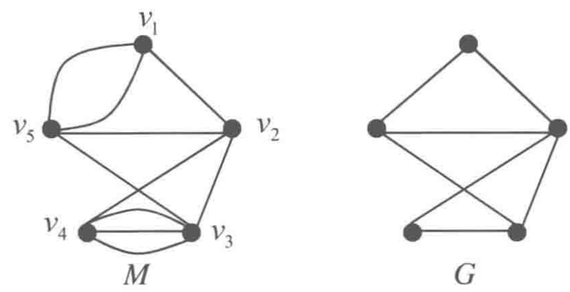

Figure 0.1: A multigraph *M* and a graph *G*.

Note that a multigraph may have several geometric representations that look different. We say that the multigraphs $M'=(V',m')$ and $M''=(V'',m'')$ are *isomorphic* if there is a bijection $f:V'\to V''$ such that for every $\{v_1,v_2\}\in V'_{(2)}$, $m'(\{v_1,v_2\})=m''(\{f(v_1),f(v_2)\})$. The two multigraphs depicted in Figure 0.2 are isomorphic, under the bijection $f(v_i)=u_i$.

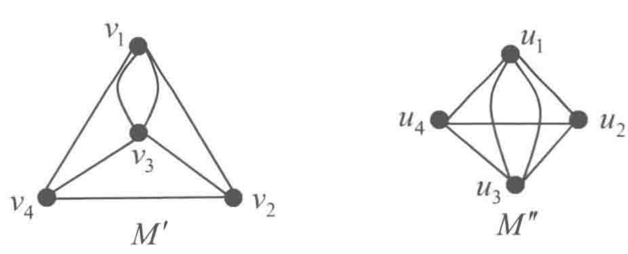

Figure 0.2: Two isomorphic multigraphs.

Edges are *incident* at its end-nodes, and two nodes which have an edge between them are called *adjacent*. Node u has *degree* $\deg(u)$ equal to the number of edges having that node as an end-node, i.e. $\deg(u) = \sum_{v \neq u} m(\{u,v\})$.

The following theorem, the First Theorem of Multigraph Theory, was proved by Euler in 1736.

Theorem 0.1 (The First Theorem of Multigraph Theory) *The sum of the degrees of all the nodes of a multigraph M is equal to twice the number of edges.*

Proof Let $x = v_i v_j$ be an edge of M, then x contributes 1 to both $\deg(v_1)$ and $\deg(v_2)$. Thus when summing the degrees of the nodes of M, each edge is counted twice. □

Another way to represent a multigraph is by a matrix. Let $M = (V, m)$ be a multigraph of order n, with node set $V = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of M, denoted $\mathbf{A}(M)$ or simply $\mathbf{A}$ is the $n \times n$ matrix $[a_{ij}]$ where $a_{ij} = m(\{v_i, v_j\})$. Note if $\mathbf{A}$ is the adjacency matrix of M, then $\sum_{j=1}^{n} a_{ij} = \deg(v_i)$.

Example 0.1 The adjacency matrix of the multigraph in Figure 0.1 is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 1 & 3 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

A multigraph M is *r-regular* if each node has degree equal to r. When the specific value of r is not needed we say the multigraph is *regular*.

The multigraph $M' = (V', m')$ is called a *submultigraph* of $M = (V, m)$ if $V' \subseteq V$ and $m'(\{u,v\}) \le m(\{u,v\})$ for all $\{u,v\} \in V'_{(2)}$. If $V' = V$, then M' is called a *spanning submultigraph*. If M is a graph then we use the terms *subgraph* and *spanning subgraph*. If $m'(\{u,v\}) = m(\{u,v\})$ for all $\{u,v\} \in V'_{(2)}$, then $M' = (V', m')$ is an *induced submultigraph* and is denoted by $\langle V' \rangle$. If $\langle V' \rangle$ contains no edges, i.e. $m(\{u,v\}) = 0$ for all $\{u,v\} \in V'_{(2)}$ then V' is called an *independent set of nodes*. If $V' = V$ and $m'(\{u,v\}) = \min\{m(\{u,v\}), 1\}$ for all $\{u,v\} \in V'_{(2)}$, then $M' = (V', m')$ is a spanning subgraph called the *underlying graph of the multigraph* (V, m). In Figure 0.1 G is the underlying graph of multigraph M.

A *path* in a multigraph M is an alternating sequence of nodes and edges $v_1, x_1, v_2, x_2, v_3, \ldots, x_{k-1}, v_k$, where $\{v_1, v_2, \ldots, v_k\}$ are distinct nodes and x_i is an edge between v_i and v_{i+1}. If the multigraph is in fact a graph then it is only necessary to list the nodes. The *length of a path* is the number of edges in the path. It is evident from the definition that a path on k nodes has length $k-1$. A *cycle* in a multigraph M is an alternating sequence of nodes and edges $v_1, x_1, v_2, x_2, v_3, \ldots, x_{k-1}, v_k, x_k, v_1$, where $\{v_0, v_1, \ldots, v_k\}$ are distinct nodes and x_i is an edge between v_{i-1} and v_i, $1 \le i \le k-1$, and x_k is an edge between v_k and v_0. The *length of a cycle* is the number of edges in the cycle, so a cycle on *k* nodes has length *k*. In a multigraph it is possible to have cycles of length 2, i.e. if there are multiple edges between a pair of nodes, but in a graph the minimum cycle length is 3. A graph *G* is *acyclic* if it has no subgraphs which are cycles.

A multigraph is *connected* when every partition of the node set $V = V_1 \cup V_2$, $V_1, V_2 \neq \varnothing$, and $V_1 \cap V_2 = \varnothing$ has at least one multiedge with one endpoint in V_1 and the other in V_2. Alternately, a multigraph is connected if there is at least one path between every pair of nodes. A multigraph which is not connected is *disconnected*. A *component* of a multigraph *M* is a maximal connected submultigraph M', i.e. if M'' is a submultigraph of *M* that properly contains M', then M'' is disconnected. A disconnected multigraph contains at least two components, thus a multigraph is connected if and only if it has one component.

A *tree* is a connected graph with n nodes and $n-1$ edges. Considering a longest path in a tree it is easy to see that a tree has at least two nodes of degree 1, called *leaves* or *pendant* edges. A spanning subgraph that is also a tree is called a *spanning tree*. Figure 0.3 depicts a multigraph and one of its spanning trees. It is easy to see that a multigraph has spanning trees if and only if it is connected.