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Kac-Moody Algebras and Their Representations

Xiaoping Xu

(卡茨－穆迪代数及其表示)

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In order to study infinite-dimensional Lie algebras with root space decomposition as finite-dimensional simple Lie algebras, Victor Kac and Robert Moody independently introduced Lie algebras associated with generalized Cartan matrices, so-called "Kac-Moody algebras " in later 1960s. In last near forty years, these algebras have played important roles in the other mathematical fields such as combinatorics, number theory, topology, integrable systems, operator theory, quantum stochastic process, and in quantum field theory of physics. This book gives a systematic exposition on the structure of Kac-Moody algebras, their representation theory and some connections with combinatorics, number theory, integrable systems and quantum field theory in physics. In particular, we give many details that Kac's book lacks, correct some mistakes and reorganize the materials. This book may serve as a text book for graduate students in mathematics and physics, and a reference book for researchers.

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Dedicated to My Parents

Preface

In order to study infinite-dimensional Lie algebras with root space decomposition as finite-dimensional simple Lie algebras, Victor Kac and Robert Moody independently introduced Lie algebras associated with generalized Cartan matrices, so-called "Kac-Moody algebras" in later 1960s. In last near forty years, these algebras have played important roles in the other mathematical fields such as combinatorics, number theory, topology, integrable systems, operator theory, quantum stochastic process, and in quantum field theory of physics.

There have been several books on Kac-Moody algebras. The most authoritative and influential one may be the monograph "Infinite-dimensional Lie algebras" by Victor Kac. Our purpose of writing this book is to help the readers to better understand Kac's book. This book was written based on my lecture notes in Kac-Moody algebras taught in Chinese Academy of Mathematics and System Sciences in 2005, 2006. We have tried to give the details that Kac's book lacks and correct some mistakes. In many occasions, we have reorganized the materials. For instance, we have added detailed vertex operator and free fermionic field representations of affine Kac-Moody algebras. Of course, we have also deleted some materials in Kac's book which do not seem so important to students. Nevertheless, our book contains most of fundamental results in Kac-Moody algebras. Needless to say, this book is not a replacement of Kac's book, but a new choice of textbooks in the field to researchers and students.

I would like to thank my friends Prof. Shaobin Tan and Prof. Yucai Su for their encouragement of writing this book. I am also very grateful to my students Li Luo and Yufeng Zhao, and to Yan Wang (a graduate student at Nan Kai University) for their careful proof reading of the initial manuscript and pointing out numerous typos and errors.

Xiaoping Xu
2006, Beijing

Notational Conventions

$\mathbb{C}$	the field of complex numbers.
$\overline{i, i+j}$	$\{i, i+1, i+2, \cdots, i+j\}$, an index set.
$\delta_{i,j} = 1$ if $i = j$, 0 if $i \neq j$.	
$\mathbb{Z}$	the ring of integers.
$\mathbb{Z}_+$	$\{0, 1, 2, 3, \cdots\}$, the set of natural numbers
$\mathbb{Q}$	the field of rational numbers.
$\mathbb{R}$	the field of real numbers.
Π	the set of positive simple roots.
Δ	the set of roots.
Ω	generalized Casimier operator
$W(A)$	the Weyl group.
r_i	the ith simple reflection.

Contents

Introduction

Mathematics is a logical science. Lie algebra is not a mysterious subject. It can be viewed as "advanced linear algebra" in a certain sense. In linear algebra, we mainly study vector spaces and single linear transformation. Recall that an $n \times n$ Jordan block is a matrix of the form:

$$J_{n,\lambda} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \lambda & \ddots & 0 \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}, \tag{0.1}$$

where $\lambda \in \mathbb{C}$, the field of complex numbers. A fundamental theorem in linear algebra says that any linear transformation T on a finite-dimensional vector space over $\mathbb{C}$ takes the *Jordan form* :

$$T = \begin{pmatrix} J_{n_1,\lambda_1} & 0 & \cdots & 0 \\ 0 & J_{n_2,\lambda_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{n_k,\lambda_k} \end{pmatrix} \tag{0.2}$$

with respect to a certain basis.

Lie algebra is a field of studying a vector space V and a subspace $\mathcal{G}$ of linear transformations on V such that

$$AB - BA \in \mathcal{G} \qquad \text{if } A, B \in \mathcal{G}, \tag{0.3}$$

where $\mathcal{G}$ is called a *Lie algebra* and the linear transformation $AB - BA$ is called the *commutator* of linear transformations, denoted as $[A, B]$. "Simple Lie algebra" and "irreducible modules" in Lie theory are generalizations of the Jordan blocks. "Completely reducibility" is exactly a generalization of the Jordan form.

Lie algebra is not purely an abstract mathematics but a fundamental tool of studying symmetries in the world. In fact, Norwegian mathematician Sorphus Lie introduced Lie algebra in later 19th century in order to study the symmetry of differential equations. For instance, we have the following theorem on a system of ordinary linear differential equations:

Theorem of Lie and Scheffers *The general solution $\vec{x}(t)$ of the system of equations:*

$$\frac{dx_i(t)}{dt} = f_i(\vec{x}, t), \qquad i = 1, 2, \cdots, n, \tag{0.4}$$

can be expressed as a function of m particular solutions and n significant constants

$$\vec{x}(t) = \vec{s}(\vec{x}_1(t), \cdots, \vec{x}_m(t), c_1, \cdots, c_n) \tag{0.5}$$

if and if only

$$f_i(\vec{x}, t) = \sum_{j=1}^{m} \xi_{i,j}(\vec{x}) g_j(t), \tag{0.6}$$

and the differential operators

$$\left\{ \sum_{i=1}^{n} \xi_{i,j}(\vec{x}) \partial_{x_i} \mid j = 1, 2, \cdots, m \right\} \tag{0.7}$$

span a Lie algebra of dimension m with respect to the commutator.

In general, a differential equation can be solved explicitly just because it has a certain symmetry related to Lie algebras.

Lie algebras are the infinitesimal structures (bones) of Lie groups, which are symmetric manifolds. Stochastic Leowner evolution is connected to Lie algebras with one-variable structure via conformal field theory (cf. [HP], [LSW], [So]). The controllability property of the unitary propagator of an N-level quantum mechanical system subject to a single control field can be described in terms of the structure theory of semisimple Lie algebras (cf. [DPRR]). Moreover, Lie algebras were used to explain the degeneracies encountered in the genetic code as the result of a sequence of symmetry breakings that have occurred during its evolution (cf. [HH]).

The initial of quantum physics is the *uncertainty principle*, which says that one can not measure the momentum and position of a particle at the same time. If we denote by ΔP the error of the momentum and by Δx the error of position, then

$$\Delta P \cdot \Delta x > \hbar, \tag{0.8}$$

where $\hbar$ is called the *Plunk constant.* Let $\mathbb{C}[x]$ be the algebra of polynomials in x. Define the left multiplication operator $L_x : \mathbb{C}[x] \to \mathbb{C}[x]$ by

$$L_x(f(x)) = xf(x) \qquad \text{for } f(x) \in \mathbb{C}[x]. \tag{0.9}$$

Represent the momentum by the operator d/dx and the position by L_x. The uncertainty can be mathematically interpreted as that the two operators d/dx and L_x can not have a common eigenvector, due to the non-commutativity:

$$\frac{d}{dx} \circ L_x - L_x \circ \frac{d}{dx} = \mathrm{Id}_{\mathbb{C}[x]}. \tag{0.10}$$

In quantum physics, a physical entity becomes an operator on a certain Hilbert space of physical states, which are probability functions. The Hilbert space are usually symmetric with respect to a certain Lie algebra. A *quantum field* is an operator-valued function on the Hilbert space. It turns out that the coefficients of certain quantum fields are connected to "affine Kac-Moody algebras". Throughout this book, all the vector spaces and algebras are assumed over $\mathbb{C}$.

For a vector space V, we denote by V^* the space of linear functions on V. The motivation of introducing Kac-Moody algebra was essentially to study the following type of Lie algebra $\mathcal{G}$: (1) $\mathcal{G}$ contains a subspace H such that

$$\mathcal{G} = \bigoplus_{\alpha \in H^*} \mathcal{G}_\alpha, \qquad \mathcal{G}_\alpha = \{u \in \mathcal{G} \mid [h, u] = \alpha(h)u \text{ for } h \in H\}, \tag{0.11}$$

and $\mathcal{G}_0 = H$; (2) $\bigoplus_{0 \neq \alpha \in H^*} \mathcal{G}_\alpha$ is contained in the subalgebra $\mathcal{G}'$ generated by $\{\mathcal{G}_{\pm\alpha_1}, \mathcal{G}_{\pm\alpha_2}, \cdots, \mathcal{G}_{\pm\alpha_n}\}$ with $\dim \mathcal{G}_{\pm\alpha_i} = 1$; (3) the subalgebra $\mathcal{G}_+$ generated by $\{\mathcal{G}_{\alpha_1}, \mathcal{G}_{\alpha_2}, \cdots, \mathcal{G}_{\alpha_n}\}$ dose not contain a nonzero proper subspace U such that

$$[u, v] \subset U \qquad \text{for } u \in U, v \in \mathcal{G} \tag{0.12}$$

and neither does the subalgebra $\mathcal{G}_-$ generated by $\{\mathcal{G}_{-\alpha_1}, \mathcal{G}_{-\alpha_2}, \cdots, \mathcal{G}_{-\alpha_n}\}$. Such an idea was also used by Li and the author [LX] to characterize "lattice vertex operator algebras".

A Lie algebra $\mathcal{G}$ is called *simple* if it does not contain a nonzero proper subspace U such that

$$[u, v] \in U \qquad \text{for } u \in U, \ v \in \mathcal{G}. \tag{0.13}$$

The works of Killing and Cartan showed that any finite-dimensional simple Lie algebra is of the above type. For a Lie algebra $\mathcal{G}$ and $u \in \mathcal{G}$, the *adjoint operator* ad u is defined by

$$(\mathrm{ad}\, u)(v) = [u, v] \qquad \text{for } v \in \mathcal{G}. \tag{0.14}$$

Chevalley proved that a finite-dimensional simple Lie algebra $\mathcal{G}$ has generators $e_i \in \mathcal{G}_{\alpha_i}$, $f_i \in \mathcal{G}_{-\alpha_i}$ and $h_i \in H$ for $i = 1, 2, \cdots, n$ such that

$$[h_i, h_j] = 0, \ \ [h_i, e_j] = a_{i,j} e_j, \ \ [h_i, f_j] = -a_{i,j} f_j, \ \ [e_i, f_j] = \delta_{i,j} h_j, \tag{0.15}$$

$$(\mathrm{ad}\, e_i)^{1-a_{i,r}}(e_r) = 0, \quad (\mathrm{ad}\, f_i)^{1-a_{i,r}}(f_r) = 0, \qquad i \neq r, \tag{0.16}$$

where the matrix $A = (a_{i,j})_{n\times n}$ is the *Cartan matrix* whose entries are integers satisfying:

$$a_{i,i} = 2, \;\; a_{i,j} \leqslant 0, \;\; a_{i,j} = 0 \Leftrightarrow a_{j,i} = 0, \quad i \neq j. \tag{0.17}$$

Serre [Sj] showed that (0.15) and (0.16) are indeed the defining relations of $\mathcal{G}$.

An $n \times n$ integer matrix A satisfying (0.17) was called a *generalized Cartan matrix* by Kac [Kv1] and Moody [Mr1]. Moreover, they used (0.15) and (0.16) to construct a Lie algebra $\mathcal{G}(A)$ of the type indicated earlier, so-called *Kac-Moody algebra*. The matrix A is called *symmetrizable* if there exists an invertible diagonal matrix D such that DA is symmetric. The whole business in Kac-Moody algebras took off under the assumption that A is symmtrizable. Under this assumption, the Lie algebra $\mathcal{G}(A)$ has a nondegenerate invariant bilinear form, by which a generalized Casimier operator is obtained. The structure and highest-weight representation theories were established by using the generalized Casimier operator and Weyl group. In particular, an analogue of the Weyl character formula was obtained by Kac [Kv2] for integrable highest-weight irreducible modules.

The fundamental difference between Kac-Moody algebras of infinite type and finite-dimensional simple Lie algebras is that there are roots which are not conjugated to simple roots, that is, *imaginary roots*. It turns out that imaginary roots are exactly the elements in the root lattice with non-positive square norm. The restriction of a highest-weight modules to an imaginary root subalgebra is isomorphic to a direct sum of its Verma modules. When A is indecomposable and of co-rank one, the algebra $\mathcal{G}(A)$ is called an *affine Kac-Moody algebra*. It turns out that affine Kac-Moody algebras have natural loop-algebra realizations, which imply by Sugawara operators that they are conformal invariant. In fact, they appeared in physics as "current algebras". The denominator identities of affine Kac-Moody algebras are exactly the well-known Macdonald's identities (cf. [Mi]). The characters of integrable highest-weight modules of affine Kac-Moody algebras satisfy certain modular transformation properties. In particular, the q-dimensions of the subspaces of imaginary root strings in the modules are modular forms. Affine Kac-Moody algebras are also symmetries of some integrable systems.

Lepowsky and Wilson [LW1] introduced vertex operators in order to study the explicit structure of integrable highest-weight modules of affine Kac-Moody algebras. They [LW2, LW3] used these operators to prove the famous Rogers-Ramanujan identities. Frankel [Fi], and Feingold and Frenkel