Electrons in Strong Electromagnetic Fields: An Advanced Classical and Quantum Treatment

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Preface

This book presents a quantum-theoretical consideration of some important and interesting processes occurring in the presence of intense electromagnetic fields in both the quasiclassical and essentially quantum regions of an electron motion. These processes must be investigated using "the exact solutions method" of relativistic wave equations in external electromagnetic fields. This method enables one to study different non-linear (in external field strengths squared) effects, for which the conventional treatments, namely, by the methods of perturbation theory, are hardly applicable; it may be particularly useful in studying problems analytically.

For studying the main thermal properties of a relativistic electron gas in a quantizing magnetic field the powerful method of Mellin's transformations (MT) is applied; it appears to open highly promising prospects in the physics of charged particles, and other related areas. I believe that methods applied in the book will be useful for undergraduate students, graduate students and scientific workers who specialize in the field of theoretical physics.

In writing this book the aim was to give special consideration to some important physical effects that may occur when the interactions of electrons with external electromagnetic fields are strong enough and in such cases the methods of perturbation theory are hardly applicable to study such effects. The use of "the exact solutions method" to solve various particle-interaction problems enables one not only to describe them as precisely as possible but also to see several important heretofore unknown features of the effects.

The considered effects are interesting in their own right. It is also important to know properties of the vacuum and matter under extreme conditions in connection with various attempts to formulate a theory of astrophysical objects surrounded by superstrong magnetic fields. Theoretical description of some of the effects observed under such conditions and comparison of the obtained results with astrophysical investigations make it possible to verify conclusions of the charged particle interaction theory in strong electromagnetic fields. This verification is rather interesting because the laboratory realization of such superstrong fields is not yet possible. In recent years, there has been much interest in solid state physics, investigations of quantum processes with electrons in quantizing (electron motion) magnetic fields.

The book is mainly based on calculations and works of myself and my colleagues. I have tried to present the material in such a manner that readers could carry out (or repeat) corresponding calculations using only the book. This is why the bibliography is by no means complete: it should be used as material that is necessary for understanding the problems considered. Only the best known monographs, textbooks and papers that deal closely with the problems discussed have been cited. I offer my apologies to authors of papers concerning the problems considered in this book whose works have not been cited for the above reason. A book, *Electrons in a Strong Magnetic Field*, appeared in a Russian edition (Energoatomizdat) in 1988. This book is not merely a mechanical translation in English of that earlier book. All chapters of *Electrons in a Strong Magnetic Field* have been essentially revised, and many original materials have been included in this book.

For the convenience of readers all chapters of the book are introduced with brief abstracts, introductions to the problems, and (sometimes) conclusions. I would like to express my gratitude to Dr O. Dorofeyev, and Dr G. Chizhov for their assistance and for many useful discussions during the writing of the book. Their valuable remarks and suggestions were always helpful. Dr O. Dorofeyev also made some numerical calculations (using a computer) which enabled a number of figures to be plotted. I would like to thank Prof. V.N. Rodionov who kindly agreed to write Chapter 9.

I would also like to thank Dr O. Dorofeyev and Dr G. Chizhov for their assistance in preparing the manuscript for publication.

Notations and Metric

The Metric we employ yields the following expression for the scalar (dot) product of two four-vectors

$$(kx) \equiv k \cdot x = \mathrm{k}_{\mu} x^{\mu} = k^0 x^0 - k^1 x^1 - k^2 x^2 - k^3 x^3.$$

Four-vectors k_μ are written in light face type, three-vectors $\boldsymbol{k}$ are written in black face tape. The normal three-space components of a vector are the contravariant components of the four-vector

$$\boldsymbol{k} = (k^1, k^2, k^3) = (k_x, k_y, k_z).$$

The scalar product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written as

$$\boldsymbol{a} \cdot \boldsymbol{b} = a^1 b^1 + a^2 b^2 + a^3 b^3$$

and the vector (cross) product is written by $(\boldsymbol{a} \times \boldsymbol{b})$.

The product of the Dirac matrices γ_μ and four-vector a^μ is designed as

$$\gamma \cdot a \equiv \gamma^\mu a_\mu = \hat{a}.$$

The Greek indices and the Latin indices run through the values 0, 1, 2, 3, and 1, 2, 3, respectively.

As a rule we employ a system of units in which the Dirac (action) constant $\hbar$ and the velocity of light c are equal to zero, except when specific units must be attached to a result.

We will generally use these notations: $\alpha \equiv e^2$ — for the fine-structure constant, E and ε — for the energy of particles, $\boldsymbol{E}$ — for the electric field strength, $\boldsymbol{H}$ and $\boldsymbol{B}$ — for the magnetic field strength, or for the magnetic field induction, $e = -|e|$ — for the electric charge. and m for the mass of an electron.

Contents

CHAPTER 1

Semiclassical Spin Effects in Strong Coulomb and Magnetic Fields

Abstract

A new interpretation of the well-known semiclassical phenomenon discovered by A.Sokolov and I.Ternov, "the radiative polarization" (or "the self-polarization") of electrons and positrons due to synchrotron radiation, is proposed. Special consideration is given to the physical nature of this important (for high energy physics) phenomenon from the point of view of quantum electrodynamics. The possibility that radiative corrections in electron energy may produce this effect is investigated. The electron–positron pair production by the Coulomb field of heavy nuclei located in a superstrong magnetic field is studied in detail. It is shown that, in the presence of a superstrong magnetic field and when $Z_{cr}|e|$ (where $Z_{cr}|e|$ is the critical electric charge of a nucleus) approximately Z positrons are created, which are polarized along the magnetic induction vector, and that the vacuum electron shell formed has a small magnetic moment. Thus, in the case under discussion, the vacuum of quantum electrodynamics must have a macroscopic electric charge as well as a macroscopic magnetic moment. New and very interesting peculiarities reveal themselves due to the screening of the Coulomb field of the nucleus by the electron shell formed, in particular, the distribution of the vacuum (electric) charge in ultraheavy nuclei located in a superstrong magnetic field. A relativistic analog to the Thomas–Fermi equation in the presence of a superstrong magnetic field is deduced to describe the distribution of electrons in atoms as well as the charge distribution in the vacuum electron shell of nuclei.

1. RADIATIVE POLARIZATION OF AN ELECTRON IN MAGNETIC FIELDS

1.1. Radiative Polarization of Electrons

In our opinion there occurs at least one interesting effect in the semiclassical region of electron motion that is very important for high-energy physics. This effect, predicted by A.Sokolov and I.Ternov in 1963, which they called the "self-polarization effect" of electrons, is due to the response reaction of photon emission by electrons moving in a uniform constant magnetic field. It was subsequently confirmed by theoretical studies of Bayer (1967) and Schwinger (1975). A large number of experiments were carried out in France, Italy, Russia, Germany, and the USA. Their results were mainly consistent with theoretical predictions. We would like to give attention to this important physical phenomenon, with an emphasis on its quantum electrodynamical origin.

The quantum state of a free electron can be described by the function

$$\psi_p = \frac{1}{(2E)^{1/2}} u_p \exp(-ipx), \tag{1.1}$$

where E is the energy of an electron. The plane wave amplitude u_p is a spinor playing the role of the spin wave function, p is the four-dimensional momentum. It is known [2, 14 17, 116, 117, 120] that the spin electron can be characterized by the spin (pseudovector) operator σ_μ, whose space-like components in turn are determined by the (pseudovector) σ. However, the spin pseudovector for the Dirac equation is not an integral of motion and this does not enable one to use its eigenvalues together with the Hamiltonian ones. In other words these operators do not have common wave functions and therefore the spin projection of an electron in some arbitrary direction (for example along the OZ axis) cannot have a definite value at given three-dimensional momentum $\boldsymbol{p}$. This follows from the Hamiltonian of the Dirac electron

$$\mathcal{H} = (\alpha \cdot \boldsymbol{p}) + \rho_3 m. \tag{1.2}$$

Here m is the mass of an electron (it is well to remember that σ, α and ρ_3 here are the Dirac four-by-four matrices), which with given momentum $\boldsymbol{p}$ does not commute with the matrix operator σ_z. Nevertheless the Hamiltonian commutes with the matrix $(\boldsymbol{n} \cdot \sigma)$ in which $\boldsymbol{n} = \boldsymbol{p}/|\boldsymbol{p}|$. Hence the spin projection of the electron in the direction of electron motion is conserved. This is the electron helicity.

For solving physical problems we need to find the Dirac equation solutions as functions of all quantum numbers of the conserved operators. There are four functionally-independent operators and one of them describes the electron spin. To construct a conserved spin operator we need to take into account two more conditions. As a rule such operators are differential operators of the first order, i.e. they are linear in the electron momentum in the momentum representation; the momentum coefficients are the Dirac matrices not proportional to the unit matrix. To be more exact, these matrices contain the vector matrix σ. In the nonrelativistic approximation spin operators must be proportional to two-column Pauli spin matrices.

Then operators constructed can be generalized for the case of electron motion in an external electromagnetic field. This may be done by means the "principle of minimum electromagnetic interaction". For this we need to replace the four-momentum operator p by the generalized four-momentum operator P as follows

$$P^{\mu} = p^{\mu} - eA^{\mu}_{\text{ext}}. \tag{1.3}$$

It is as well to note that, after substitution of P^{μ} for p^{μ} into the Dirac equation, the symmetry of the Hamiltonian, which now describes an electron in the presence of external fields, may be broken. It will occur in the case in which the four-potential of the external field A_{ext} depends on any spatial coordinates or time, for which reason only a single component of the spin operator may remain an integral of motion in the presence of a given electromagnetic field.

The main theoretical conclusion was formulated in [115] for a bunch of electrons because it was made on the basis of solutions of kinetic equations. The self-polarization effect for many electrons is that a bunch of electrons moving in an external uniform magnetic field for a long time must be partially polarized. It is extremely important that the energy of electrons be kept constant for the so-called polarization time. Since this time is long, observation of the effect under consideration is possible only in storage rings.

There appears one more question, about a bunch in which there is only a single electron. Put another way, we want to study the behavior of the electron spin but not the bunch polarization. This problem was solved by Bayer.

One describes the electron spin in several ways. The way used to describe it in [115] was in terms of the z-projection of the well-known spin operator μ

$$\mu = \sigma + \rho_2 \frac{(\sigma \times \boldsymbol{P})}{m}, \tag{1.4}$$

since it is known that it is an integral of motion in a constant uniform magnetic field $\boldsymbol{H} = (0,\ 0,\ H)$. The operator μ_z characterizes the spin states with transverse polarization when the momentum projection $p_z = 0$.

The probability of an emission transition from the initial state n, ζ to the state $n' < n$, ζ' summed over all quantum numbers except for ζ, ζ' and averaged over two possible states of photon polarization, in which quantities of the order of χ^2 are conserved, was found in [115, 116] in the form

$$w_{\zeta\zeta'} = \frac{5\sqrt{3}}{6} e^2 \frac{eH}{m} \left\{ \frac{1+\zeta\zeta'}{2} \left[1 - \frac{8\sqrt{3}}{15}\chi + \frac{25}{8}\chi^2 - \frac{3\zeta}{10}\chi \left(1 - \frac{10\sqrt{3}}{3}\chi \right) \right] \right.$$

$$\left. + \frac{1+\zeta\zeta'}{2} \frac{3}{8}\chi^2 \left(1 + \zeta \frac{8\sqrt{3}}{15} \right) \right\}, \tag{1.5}$$

where e is the charge of an electron in absolute value,

$$\chi = \frac{\left(e F_{\mu\nu}^{\text{ext}} p^{\nu} \right)^2}{m^3} \equiv \frac{|e| EH}{m^3}$$

is the relativistic invariant dynamic parameter, in which $F_{\mu\nu}^{\text{ext}}$ is the tensor of external electromagnetic field, p is the four-momentum of electron. Quantities ζ, ζ' appearing in (1.5) are the eigenvalues of the μ_z-spin operator normalized to unity.

The ratio of radiative transition probability at which the electron spin orientation changes $(\zeta = -\zeta')$ to that at which it is conserved $(\zeta = \zeta')$ is

$$w_{\zeta=-\zeta'} / w_{\zeta=\zeta'} \cong \chi^2, \tag{1.6}$$

where in turn

$$w_{\zeta=\zeta'} \cong |e|^3 H / m. \tag{1.7}$$

It follows from formula (1.5) that the probability of quantum transition of the electron per unit time from the state with $\zeta = 1$ (the electron spin and the vector $\boldsymbol{H}$ have the same direction) to the state with $\zeta' = -1$ (the electron spin direction is opposite to the magnetic field strength) is greater than that from the state $\zeta = -1$ to the state $\zeta' = 1$. It will be noted that the effect is revealed to be very small under usual conditions. The ratio of transition probabilities with the spin-flip and without spin-flip is of the order χ^2, which for typical laboratory conditions now is of order 10^{-12}. So the quantum transition of an electron with the spin-flip takes place approximately 10^{12} times as slowly as one when $\zeta = \zeta'$. Thus, in order for the effect to be observable, the electron has to be returned from the final quantum state to the initial state in some way or other but in such a manner that its spin projection would not be changed. Besides, it is necessary to compensate the radiative loss of electron energy by keeping the electron energy constant for a time about 10^{12} s. This is done by a high-frequency electric field that cancels out the radiative energy loss in storage rings. It enables one not to consider at all the quantum transitions without spin-flip and, in what follows, to solve a more simple (but equivalent) problem of the behavior of some two-level systems. Let us call it the "spin-system". We do not have complete information about this system to describe it in terms of a definite wave function because this system may go through the transitions due to the interaction with the second-quantized photon field but not due to the interaction with the external (classical) electromagnetic field. Since the probabilities of these transitions can be calculated, one knows that such a kind of system may be described by the density matrix. It occurred to Bayer to use the density matrix in order to describe this system [14].

Let $P_{\pm 1}$ be the probabilities determining the spin system in the eigenstates with $\zeta = \pm 1$. Then, according to nonstationary perturbation theory, we can write the following equations

$$\frac{dP_1}{dt} = w_{1,-1}P_{-1} - w_{-1,1}P_1, \tag{1.8}$$

$$\frac{dP_{-1}}{dt} = w_{-1,1}P_1 - w_{1,-1}P_{-1}. \tag{1.9}$$

It follows from (1.8) and (1.9) that

$$\frac{d(P_1 + P_{-1})}{dt} = 0,$$

$$P_1 + P_{-1} = 1.$$

The probabilities introduced thus are the diagonal elements of the density matrix, since, according to the definition of the density matrix, its diagonal element ρ_{ii} is the probability for the detection of the system in the eigenstate with given number i. The solutions of (1.8) and (1.9) have the simple forms

$$P_1(t)=\frac{w_{1,-1}}{w_{1,-1}+w_{-1,1}}-Ce^{-(w_{1,-1}+w_{-1,1})t}, \quad (1.10)$$

$$P_{-1}(t)=\frac{w_{-1,1}}{w_{1,-1}+w_{-1,1}}+Ce^{-(w_{1,-1}+w_{-1,1})t}. \quad (1.11)$$

It will be noted that, in the limit $t>>\tau=(w_{1,-1}+w_{-1,1})^{-1}$, the solutions (1.10) and (1.11) are independent of the initial values $P_1(0)$ and $P_{-1}(0)$:

$$P_1(t>>\tau)=\tau w_{1,-1},$$
$$P_{-1}(t>>\tau)=\tau w_{-1,1}. \quad (1.12)$$

The parameter τ is called the polarization time. Let

$$P_1(0)=P_{10},$$

$$P_{-1}(0)=1-P_{10}=P_{-10}, \quad (1.13)$$

then

$$P_1(t)=\tau w_{1,-1}-\tau(w_{1,-1}-P_{10})e^{-t/\tau}, \quad (1.14)$$

$$P_{-1}(t)=\tau w_{-1,1}+\tau(w_{1,-1}-P_{10})e^{-t/\tau}. \quad (1.15)$$

The mean value of the normalized operator $\tilde{\mu}_z$ at the time t is determined by

$$\langle\tilde{\mu}_z(t)\rangle=P_1(t)-P_{-1}(t) \quad (1.16)$$

and at $t>>\tau$ one can obtain [115, 116]

$$<\tilde{\mu}_z(t)>\Big|_{t>>\tau}=\frac{w_{1,-1}-w_{-1,1}}{w_{1,-1}+w_{-1,1}}=-\frac{8\sqrt{3}}{15}=-0.924. \quad (1.17)$$

Here

$$\rho_{11} = w_{1,-1}/(w_{1,-1} + w_{-1,1}) = 0.96 \tag{1.18}$$

is the probability of the detection of the spin system in the eigenstate with $i = 1$, and

$$\rho_{-1-1} = w_{-1,1}/(w_{1,-1} + w_{-1,1}) = 0.04 \tag{1.19}$$

is the probability for the spin system to be detected in the eigenstate with $i = -1$.

Hence the radiative transitions of an electron in a constant magnetic field when its energy is kept constant for the time $t >> \tau$ may be described by the spin density matrix. The radiative polarization effect evidently depends on the asymmetry of the transition probabilities $w_{1,-1}$ and $w_{-1,1}$.

1.2. On the Interpretation of the Radiative Polarization Effect

The polarization state of an electron may be defined by the four-vector a^{μ}, which coincides with the three-dimensional unit pseudovector ζ in the resting system of the electron. We remind the reader that the pseudovector ζ equals the mean value of the operator of the spin vector in the resting system of an electron. One knows that the modulus of the vector ζ equals 1 if the spin state of the electron is a pure one in the sense of quantum mechanics, and it is less than 1 if this is not the case. Using the vector ζ enables us, in principle, to answer the question under discussion right away. Indeed, since the photon emission by an electron is one of the processes described by quantum electrodynamics, its probability must be G even (G is the parity operator). So, if $\zeta = 0$ in the initial quantum state, and the modulus of the vector ζ is less than 1 for the final quantum state, then the expression for the probability of such processes may contain terms linear in ζ' only as a scalar product $(\zeta' \cdot \boldsymbol{H})$ in which $\boldsymbol{H}$ is also a pseudovector, for example $\boldsymbol{H}$ is the magnetic field strength.

It is known [17] that the vector ζ characterizes the properties of the detector, which singles out, in fact, the spin projection of the electron in the final quantum state. The real final spin state of the electron determined by the vector ζ^{f} may not coincide with the state described by the vector ζ'. Nevertheless, if we know the probability of a quantum process as a function of ζ', we can find the polarization vector ζ^{f}. For this, let us write the total probability of radiative transitions of the electron with the spin-flip as a function of the vectors ζ and ζ'

$$w_{\zeta,-\zeta'} = \frac{5\sqrt{3}}{16}\frac{e^2}{m^2}\left(\frac{E}{m}\right)^2 \omega_0^3\left[1+\frac{2}{9}(\zeta\cdot\boldsymbol{v})(\zeta\cdot\boldsymbol{v})+\frac{8\sqrt{3}}{15}\frac{e}{|e|}\frac{(\zeta\cdot\boldsymbol{H})}{|\boldsymbol{H}|}\right], \quad (1.20)$$

where $\omega_0 = eH/E$ is the frequency of motion of the electron in a magnetic field. One sees that the coefficient in w depends on the electron energy and magnetic field strength but not on the vectors ζ and ζ', $\boldsymbol{v}$ is the instantaneous velocity of the electron. It is well to emphasize that the last formula describes the radiative polarization effect if the polarization state of the electron is defined by the vector ζ but not the operator μ_z.

On the other hand, the matrix element squared of an arbitrary process in quantum electrodynamics, M_{fi} as a function of the vectors ζ' and ζ^{f} must have the form

$$|M_{fi}|^2 \sim (1+\zeta'\cdot\zeta^{\mathrm{f}})/2.$$

Comparing the last formula with (1.20), we determine the vector ζ^{f}, which characterizes the final polarization state of the electron

$$\zeta^{\mathrm{f}} = \frac{2}{9}(\zeta\cdot\boldsymbol{v})\boldsymbol{v}+\frac{8\sqrt{3}}{15}\frac{e}{|e|}\frac{\boldsymbol{H}}{|\boldsymbol{H}|}. \quad (1.21)$$

It is seen that the first term on the right hand side of formula (1.21) characterizes the ζ^{f} vector projection in the direction of the electron motion while the second term is the one in the magnetic field direction. The latter is called the transverse polarization.

It is well to remember that the detector always determines the projection of the vector ζ^{f} on the vector ζ'. It is of interest that, when $\boldsymbol{v}$ is perpendicular to the vector $\boldsymbol{H}$, the detector, which singles out the transverse polarization of an electron, will measure the same value, equal to 0.924, irrespective of the electron polarization in the initial quantum state. In other words the electron polarization in the final state, when $\boldsymbol{v}$ is perpendicular to the vector $\boldsymbol{H}$, is always determined by the vector ζ^{f}, and the vector projection ζ' on the magnetic field direction is always equal to 0.924