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INTRODUCTION

Compactness is frequently an annoying hypothesis in differential topology. Even when one is primarily interested in a compact manifold, associated noncompact manifolds turn up, eg. the leaf space of a foliation. Also, in technical constructions, it would be helpful to be able to dispense with compactness. For example, if f is a diffeomorphism of a compact manifold, X, then it is helpful in studying the dynamics of f to regard the integers, $\mathbb{Z}$, as a discrete manifold and look at the manifold of maps, $\mathcal{C}(\mathbb{Z},X)$. Again it is compactness that prevents one from noting that Hartman's Theorem is not only related to the structural stability theorem for Anosov diffeomorphisms but is in fact a corollary of the latter because a hyperbolic linear map is an Anosov diffeomorphism of Euclidean space.

This book describes the category of metric manifolds and metric maps to which a broad class of theorems and constructions extend from the realm of compact manifolds. The category is a broad one because all paracompact manifolds admit metric structures. Metric theorems include compact theorems because a compact manifold admits a unique metric structure and with respect to it any smooth map with compact domain is a metric map. Finally, there is a sufficient abundance of metric maps that, for example, structural stability under perturbation within the family of metric maps remains useful.

Our principal tool is the atlas. Just as most elementary constructions in p.l. topology are really simplicial constructions so are most of the elementary constructions in differential topology really atlas constructions building new atlases from old. In Chapter I we review such standard constructions as products of spaces and bundles, pull back of

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bundles, etc. As well as illustrating the atlas point of view, this develops notation for future use. We also make some easy definitions suggested by the emphasis on atlases. For example, if $\mathfrak{a}_1 = \{U_\alpha, h_\alpha\}$ and $\mathfrak{a}_2 = \{V_\beta, g_\beta\}$ are two atlases then an <u>index preserving map</u> $f\colon \mathfrak{a}_1 \to \mathfrak{a}_2$ is a continuous map, f, of the underlying manifolds and an unnamed map of the index sets $\alpha \to \beta(\alpha)$ such that $U_\alpha \subset f^{-1}(V_{\beta(\alpha)})$ for all α.

In the past this primacy of the atlas has been ignored. Where the p.l. topologist has used triangulations the differential topologist has used charts or, equivalently, local coordinates. This is because a smooth manifold has a maximal atlas consisting of all smooth charts. However, the maximal atlas lacks certain natural tools possessed by other atlases less profligate in charts. What is needed is some control over the size of the transition maps of the atlas. Size means norm in some Banach space of functions. So, in Chapter II we study various function space types.

A <u>function space type</u> $\mathfrak{M}$ associates to every Banach space F and every bounded open set U in a Banach space E a Banach space $\mathfrak{M}(U,F)$ of functions from U to F. For example, $\mathfrak{B}(U,F)$ consists of bounded functions f with the sup norm $\|f\|_0 = \sup\{\|f(x)\|: x \in U\}$ and $\mathcal{C} \subset \mathfrak{B}$ is the subspace of continuous functions. Defining $\|f\|_L =$ $\sup\{\|f(x) - f(y)\|/\|x - y\|: x \neq y$ and the segment $[x,y] \subset U\}$, we get $\mathcal{L}(U,F)$ the space of functions f with $\max(\|f\|_0, \|f\|_L) = \|f\|_{\mathcal{L}} < \infty$. This is the space of bounded, uniformly locally Lipschitz functions, or equivalently, of bounded functions, Lipschitz with respect to a natural intrinsic metric on U.

We follow Palais' "axiomatic" approach [21]. Thus, we define $f \in \mathfrak{M}^r(U,F)$ if f is r times differentiable and its r-jet, $j^r(f)$, lies in $\mathfrak{M}(U, J^r(E;F))$. $\|f\|_{\mathfrak{M}^r} = \|j^r f\|_{\mathfrak{M}}$. $\mathfrak{M}^r$ is the r^{th} derived function space type of $\mathfrak{M}$. Then, after checking a function space property on basic examples like $\mathcal{C}$ and $\mathcal{L}$, we verify that it is inherited as we pass from

$\mathfrak{M}$ to $\mathfrak{M}^r$. For example, the <u>Gluing Property</u> states that if $\{U_\alpha\}$ is an open cover of U and for $f: U \to F$, $f|U_\alpha \in \mathfrak{M}(U_\alpha,F)$ with $\sup_\alpha \|f|U_\alpha\|_{\mathfrak{M}} < \infty$ then $f \in \mathfrak{M}(U,F)$ with $\|f\|_{\mathfrak{M}} = \sup_\alpha \|f|U_\alpha\|_{\mathfrak{M}}$.

Define the Banach space product, $\hat{\Pi}_\alpha F_\alpha$, of a family of Banach spaces to be the set of $\{x_\alpha\} \in \Pi_\alpha F_\alpha$ such that $\|\{x_\alpha\}\| = \sup\|x_\alpha\| < \infty$, with this sup norm. The <u>Strong Product Property</u> states that if $f_\alpha \in \mathfrak{M}(U,F_\alpha)$ and $\sup\|f_\alpha\|_{\mathfrak{M}} < \infty$ then the product map $\hat{\Pi} f_\alpha \in \mathfrak{M}(U,\hat{\Pi} F_\alpha)$ with $\|\hat{\Pi} f_\alpha\|_{\mathfrak{M}} = \sup\|f_\alpha\|_{\mathfrak{M}}$. The <u>Product Property</u> is this statement for finite index sets.

Smoothness of the composition map is handled as follows. Let $\mathfrak{M}$, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be function space types with $\mathfrak{M}_1 \subset \mathfrak{M}_2$ (i.e. $\mathfrak{M}_1(U,F) \subset \mathfrak{M}_2(U,F)$ as sets and $\| \ \|_{\mathfrak{M}_2} \le \| \ \|_{\mathfrak{M}_1}$ on the subset). We say that <u>$\mathfrak{M}_1$ maps $\mathfrak{M}$ to $\mathfrak{M}$ in an $\mathfrak{M}_2$ way</u> if whenever $U \subset F_1$, $G \subset F_2$ and $V \subset E_1$ are open and bounded and $H: G \to \mathfrak{M}(U,E_1)$ is an $\mathfrak{M}_2$ map with Image $H(g) \subset V$ for all $g \in G$, then the bounded linear map $\Omega_H: \mathfrak{M}_1(V,E_2) \to \mathfrak{M}_2(G,\mathfrak{M}(U,E_2))$ is well defined by $\Omega_H(f)(g) = f \circ H(g)$ and $\|\Omega_H\| \le O^*(\|H\|_{\mathfrak{M}_2})$. The latter is a typical estimate for our work and it means there exist constants K, n depending only on the function space types such that $\|\Omega_H\| \le K \max(\|H\|_{\mathfrak{M}_2},1)^n$. For example, that $\mathfrak{M}$ maps $\mathfrak{M}$ to $\mathfrak{M}$ in a $\mathfrak{B}$ way means that if $g: U \to V$ is an $\mathfrak{M}$ map then $g^*: \mathfrak{M}(V,E) \to \mathfrak{M}(U,E)$ is a bounded linear map well defined by $g^*(f) = f \circ g$ and $\|g^*\| \le O^*(\|g\|_{\mathfrak{M}})$.

The Gluing and Product Properties inherit in the obvious way. For the composition property there are two inheritance theorems: If $\mathfrak{M}_1$ maps $\mathfrak{M}$ to $\mathfrak{M}$ in an $\mathfrak{M}_2$ way then $\mathfrak{M}_1^r$ maps $\mathfrak{M}^r$ to $\mathfrak{M}^r$ in an $\mathfrak{M}_2$ way and if, in addition, $\mathfrak{M}_2 \subset \mathcal{C}$ then $\mathfrak{M}_1^r$ maps $\mathfrak{M}$ to $\mathfrak{M}$ in an $\mathfrak{M}_2^r$ way.

If $\mathfrak{M}$ satisfies a constellation of properties including $\mathfrak{M} \subset \mathcal{C}$, Gluing, Product (but not necessarily the Strong form) and $\mathfrak{M}$ maps $\mathfrak{M}$ to $\mathfrak{M}$ in a $\mathfrak{B}$ way then $\mathfrak{M}$ is called a <u>standard</u> function space type. Being standard is a heritible property. $\mathcal{C}$ and $\mathcal{L}$ are standard (it is to get the Gluing Property for $\mathcal{L}$ that we use the seminorm $\| \ \|_L$ rather than its more obvious Lipschitz constant relatives) and hence so are $\mathcal{C}^r$ and $\mathcal{L}^r$.

$\mathcal{L}^r$ also satisfies the Strong Product Property. Finally, $\mathcal{L}^{r+s}$ maps $\mathcal{C}^r$ to $\mathcal{C}^r$ in an $\mathcal{L}^s$ way (and so $\mathcal{C}^{r+s+1}$ maps $\mathcal{C}^r$ to $\mathcal{C}^r$ in a $\mathcal{C}^s$ way) and $\mathcal{L}^{r+s+1}$ maps $\mathcal{L}^r$ to $\mathcal{L}^r$ in an $\mathcal{L}^s$ way.

In Chapter III we return to atlases and describe the category of $\mathfrak{M}$ metric manifolds and maps for any standard function space type $\mathfrak{M}$. There is also an associated category of vector bundles but we will restrict discussion here to manifolds. All manifolds are assumed to be Hausdorff Banach manifolds.

An atlas $\mathfrak{A} = \{U_\alpha, h_\alpha\}$ on X is a <u>bounded $\mathfrak{M}$ atlas</u> if the transition maps $h_\alpha h_\beta^{-1} \in \mathfrak{M}(h_\beta(U_\alpha \cap U_\beta), E_\alpha)$ $(h_\alpha(U_\alpha)$ is open in $E_\alpha)$ and $k_{\mathfrak{A}} = \max(1, \sup\|h_\alpha h_\beta^{-1}\|_{\mathfrak{M}}) < \infty$. $\mathfrak{A}$ is an <u>$\mathfrak{M}$ atlas</u> if each point x of X has a neighborhood U such that $\mathfrak{A}|U = \{U_\alpha \cap U, h_\alpha|U_\alpha \cap U\}$ is a bounded $\mathfrak{M}$ atlas. We then define $\rho_{\mathfrak{A}}\colon X \to [1,\infty)$ by $\rho_{\mathfrak{A}}(x) = \inf\{k_{\mathfrak{A}|U}\colon U$ is a neighborhood of $x\}$. $\rho_{\mathfrak{A}}$ is clearly upper semicontinuous and it follows from the Gluing Property for $\mathfrak{M}$ that $\rho_{\mathfrak{A}}$ is bounded on an open set U of X iff $\mathfrak{A}|U$ is a bounded $\mathfrak{M}$ atlas and then $k_{\mathfrak{A}|U} = \sup \rho_{\mathfrak{A}}|U$. For example, if X is a paracompact C^r manifold and $\mathfrak{A} = \{V_\alpha, h_\alpha\}$ is a locally finite C^r atlas then choosing $\{U_\alpha\}$ an open cover with $\bar{U}_\alpha \subset V_\alpha$ it is easy to check that $\mathfrak{A}_1 = \{U_\alpha, h_\alpha\}$ is a $\mathcal{C}^r$ atlas.

Let $\mathfrak{A}_1 = \{U_\alpha, h_\alpha\}$ and $\mathfrak{A}_2 = \{V_\beta, g_\beta\}$ be $\mathfrak{M}$ atlases on X_1 and X_2. A continuous map $f\colon X_1 \to X_2$ is a bounded $\mathfrak{M}$ map $f\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ if $\mathfrak{A}_1$ is a bounded $\mathfrak{M}$ atlas and the local representatives $f_{\beta\alpha} = g_\beta f h_\alpha^{-1} \in \mathfrak{M}(h_\alpha(U_\alpha \cap f^{-1}V_\beta), E_\beta)$ and $k(f;\mathfrak{A}_1,\mathfrak{A}_2) = \sup\|f_{\beta\alpha}\|_{\mathfrak{M}} < \infty$. f is an $\mathfrak{M}$ map if it is locally a bounded $\mathfrak{M}$ map and we then define $\rho(f;\mathfrak{A}_1,\mathfrak{A}_2)(x) = \inf\{k(f|U;\mathfrak{A}_1|U,\mathfrak{A}_2)\colon U$ a neighborhood of $x\}$. ρ_f has properties analogous to $\rho_{\mathfrak{A}}$. Again if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are obtained by shrinking locally finite C^r atlases as above and $f\colon X_1 \to X_2$ is a C^r map then $f\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ is a $\mathcal{C}^r$ map. Because $\mathfrak{M}$ maps $\mathfrak{M}$ to $\mathfrak{M}$ in a $\mathfrak{N}$ way it is easy to show that if $f\colon \mathfrak{A}_1 \to \mathfrak{A}_2$ and $g\colon \mathfrak{A}_2 \to \mathfrak{A}_3$ are $\mathfrak{M}$ maps then $g \circ f\colon \mathfrak{A}_1 \to \mathfrak{A}_3$ is $\mathfrak{M}$ and

$\rho_{g\cdot f} \leq (\rho_g \cdot f) O^*(\rho_f)$.

There are two other important functions associated with an atlas $\mathfrak{a} = \{U_\alpha, h_\alpha\}$ on X. $\lambda_{\mathfrak{a}}: X \to (0,1]$ is defined by $\lambda_{\mathfrak{a}}(x) = \sup\{r \leq 1$: the ball $B(h_\alpha x, r) \subset h_\alpha(U_\alpha)$ for some U_α containing $x\}$. Clearly, $\lambda_{\mathfrak{a}}$ is lower semicontinuous. For the maximal atlas on a smooth manifold $\lambda_{\mathfrak{a}} \equiv 1$. For an $\mathfrak{M}$ atlas, $\lambda_{\mathfrak{a}}$ need not be bounded away from 0. There is a useful tension in trying to bound $\rho_{\mathfrak{a}}$ and $1/\lambda_{\mathfrak{a}}$ simultaneously.

Finally, there is a pseudometric $d_{\mathfrak{a}}$ on X. An <u>$\mathfrak{a}$-chain</u> $(x_1, \alpha_1, \ldots, \alpha_N, x_{N+1})$ is a sequence such that $x_i, x_{i+1} \in U_{\alpha_i}$ and the segment $[h_{\alpha_i} x_i, h_{\alpha_i} x_{i+1}] \subset h_{\alpha_i}(U_{\alpha_i})$ $i = 1, \ldots, N$. The length of the $\mathfrak{a}$-chain is $\Sigma_{i=1}^N \| h_{\alpha_i} x_i - h_{\alpha_i} x_{i+1} \|$. If $x, y \in X$ then $d_{\mathfrak{a}}(x,y)$ is the infinum of the lengths of all $\mathfrak{a}$-chains connecting x and y. Recall that the infinum of the empty set is ∞. Allowing ∞ as a possible value, $d_{\mathfrak{a}}$ is a pseudometric. It needn't have the topology of X. In fact, if $\mathfrak{a}$ is the maximal atlas then $d_{\mathfrak{a}}(x,y)$ is 0 or ∞ according to whether x and y do or do not lie in the same component of X. However, if $\mathfrak{M} \subset \mathcal{L}$ (eg. $\mathfrak{M} = \mathcal{C}^r$ for $r \geq 1$) and $\mathfrak{a}$ is an $\mathfrak{M}$ atlas then $d_{\mathfrak{a}}$ is a metric with topology that of X. In fact, a central result--from which the Metric Theory derives its name--is the Metric Estimate which gives an explicit local comparison between $d_{\mathfrak{a}}$ and the Banach space metric pulled back to U_α by h_α. An estimate of the size of the region on which the comparison holds and the bounds in the comparison can be computed from $\lambda_{\mathfrak{a}}$ and $\rho_{\mathfrak{a}}$. It follows that only a paracompact manifold can admit $\mathfrak{M}$ atlases with $\mathfrak{M} \subset \mathcal{L}$.

A map $\rho: X \to [1,\infty)$ is called a bound on X. We say of two bounds ρ_1 and ρ_2 on X that <u>ρ_1 dominates ρ_2</u> (written $\rho_1 \succ \rho_2$) if there exist constants K, n such that $K\rho_1^n \geq \rho_2$ on X. $\succ$ is a partial ordering with associated equivalence relation $\sim$. For example, the equivalence class containing constant functions consists of all bounds ρ with $\sup \rho < \infty$.

In essence, a metric structure on X is a choice of bound on X

which dominates the growth of everything on X. In detail, an adapted $\mathfrak{M}$ atlas is a pair $(\mathcal{G},\rho)$ with $\mathcal{G}$ an $\mathfrak{M}$ atlas and ρ a bound such that $\rho > \rho_{\mathcal{G}}$. $f: (\mathcal{G}_1,\rho_1) \to (\mathcal{G}_2,\rho_2)$ is an $\mathfrak{M}$ map of adapted $\mathfrak{M}$ atlases if $f: \mathcal{G}_1 \to \mathcal{G}_2$ is an $\mathfrak{M}$ map and $\rho_1 \succ \max(\rho_f,\rho_2 \circ f)$. Two adapted $\mathfrak{M}$ atlases are equivalent, written $(\mathcal{G}_1,\rho_1) \sim (\mathcal{G}_2,\rho_2)$ if the following equivalent conditions hold: (1) $\rho_1 \sim \rho_2$ and $(\mathcal{G}_1 \cup \mathcal{G}_2,\rho_1)$ is an adapted $\mathfrak{M}$ atlas. (2) The identity maps $1: (\mathcal{G}_1,\rho_1) \to (\mathcal{G}_2,\rho_2)$ and $1: (\mathcal{G}_2,\rho_2) \to (\mathcal{G}_1,\rho_1)$ are $\mathfrak{M}$ maps. An $\mathfrak{M}$ metric structure on X is an equivalence class of adapted atlases. The atlases and bounds appearing in the structure are called admissible atlases and bounds. The admissible bounds are a bound equivalence class. In practice, we need to relate the value of ρ near x to the value at x. So we assume as part of the definition of a metric structure that there exist continuous admissible bounds. A stronger condition, which we call regularity of a metric structure, gives a uniform estimate as follows: A metric structure is regular if $\mathfrak{M} \subset \mathcal{L}$ and for some $(\mathcal{G},\rho)$ in the metric structure there exist constants K and n such that $\rho|B^{d_{\mathcal{G}}}(x,(K\rho(x)^n)^{-1}) \leq K\rho(x)^n$. This condition then holds for all $(\mathcal{G},\rho)$ in the metric structure.

If X_1 and X_2 are $\mathfrak{M}$ manifolds, i.e. manifolds with an $\mathfrak{M}$ metric structure, then $f: X_1 \to X_2$ is an $\mathfrak{M}$ map if $f: (\mathcal{G}_1,\rho_1) \to (\mathcal{G}_2,\rho_2)$ is an $\mathfrak{M}$ map for some, and hence any, choice of $(\mathcal{G}_i,\rho_i)$ in the metric structure of X_i, $i = 1,2$. In general a vectorfield, Riemannian metric or any sort of gadget is called an $\mathfrak{M}$ vectorfield, $\mathfrak{M}$ Riemannian metric or $\mathfrak{M}$ gadget if its local representatives with respect to an admissible atlas are $\mathfrak{M}$ maps the growth of whose norm is dominated by an admissible bound. We thus obtain the category of $\mathfrak{M}$ manifolds and $\mathfrak{M}$ maps with all the accoutrements of the usual differentiable category.

A regular metric manifold has a natural uniform space structure. If $(\mathcal{G},\rho)$ is in the metric structure and $m(x,y)$ is either $\max(\rho(x),\rho(y))$ or

$\min(\rho(x),\rho(y))$ then the uniformity $\mathfrak{U}_X$ is generated by the sets $\{(x,y): d_{\mathfrak{G}}(x,y) < (Km(x,y)^n)^{-1}\}$ as K and n vary over the positive integers. This uniformity is clearly finer than the uniformity associated with the metric $d_{\mathfrak{G}}$ but the two uniformities agree on bounded sets ("bounded" always means ρ-bounded). Hence, the topology associated with $\mathfrak{U}_X$ is the original topology of X. A sequence $\{x_n\}$ is $\mathfrak{U}_X$ Cauchy iff it is $d_{\mathfrak{G}}$ Cauchy and bounded. Uniform notions like uniform continuity, uniform neighborhood of a set and uniform open cover are defined via $\mathfrak{U}_X$.

A regular $\mathfrak{M}$ manifold is called semicomplete if there exist adapted atlases $(\mathfrak{G},\rho)$ in the metric structure with $\rho > 1/\lambda_{\mathfrak{G}}$. Such atlases $\mathfrak{G}$ are then called s admissible. A semicomplete manifold is uniformly (i.e. $\mathfrak{U}_X$) complete. Given uniform completeness, $\rho > 1/\lambda_{\mathfrak{G}}$ for $\mathfrak{G} = \{U_\alpha, h_\alpha\}$ iff $\{U_\alpha\}$ is a uniform open cover of X. An $\mathfrak{M}$ manifold with bounded admissible bounds is called a bounded $\mathfrak{M}$ manifold. A bounded, semicomplete manifold is called semicompact.

Dominating $1/\lambda_{\mathfrak{G}}$ is often necessary and so semicomplete manifolds are of central importance in the theory. In the theory of several complex variables such functions are already in use (eg. [18; Chap. 7]). As the name suggests it is to semicompact manifolds that many compact results generalize in the metric category. This usually happens when compactness is used in the original proof to bound functions like $\rho_{\mathfrak{G}}$ and $1/\lambda_{\mathfrak{G}}$ and to get uniformity with respect to metrics like $d_{\mathfrak{G}}$. The resulting metric theorems are true extensions of the original theorems because compact manifolds admit a unique metric structure which is semicompact (cf. Chap. VIII. Sec. 5).

In Chapter IV we apply this procedure to section spaces and manifolds of maps. Let $\pi: \mathbb{E} \to X$ be an $\mathfrak{M}$ vector bundle with admissible atlas $(\mathfrak{D},\mathfrak{G}) = \{U_\alpha, h_\alpha, \varphi_\alpha\}$. A section s of π is an $\mathfrak{M}$ section if the $(\mathfrak{D},\mathfrak{G})$ principal parts of s, defined by $\varphi_\alpha(s(x)) = (h_\alpha(x), s_\alpha(h_\alpha(x)))$, are locally $\mathfrak{M}$ maps the growth of whose $\mathfrak{M}$ norm is dominated by the

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admissible bounds on X. Thus, if π is a bounded $\mathfrak{M}$ bundle we can define the norm $\|s\|^{(\mathfrak{D},\mathfrak{G})} = \sup_\alpha \|s_\alpha\|_{\mathfrak{M}}$. This norm makes the vector space of $\mathfrak{M}$ sections of π, denoted $\mathfrak{M}(\pi)$, into a Banach space. The topology on $\mathfrak{M}(\pi)$ is independent of the atlas choice. If $(\Phi, 1_X): \pi_1 \to \pi_2$ is an $\mathfrak{M}$ linear vector bundle map of bounded $\mathfrak{M}$ bundles over X then the induced map of sections $\Phi_*: \mathfrak{M}(\pi_1) \to \mathfrak{M}(\pi_2)$ is a continuous linear map. If $f: X_0 \to X$ is an $\mathfrak{M}$ map with X_0 a bounded $\mathfrak{M}$ manifold then the pull back map $f^*: \mathfrak{M}(\pi) \to \mathfrak{M}(f^*\pi)$ is also continuous.

A <u>standard triple</u> $(\mathfrak{M}_1, \mathfrak{M}, \mathfrak{M}_2)$ is a trio of standard function space types satisfying: (1) $\mathfrak{M}_1 \subset \mathfrak{M}$ and $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \mathcal{L}$. (2) $\mathfrak{M}_1$ maps $\mathfrak{M}$ to $\mathfrak{M}$ in an $\mathfrak{M}_2$ way. (3) $\mathfrak{M}_2$ satisfies the Strong Product Condition. $(\mathcal{L}^{r+s}, \mathcal{C}^r, \mathcal{L}^s)$ and $(\mathcal{L}^{r+s+1}, \mathcal{L}^r, \mathcal{L}^s)$ are standard triples.

Let $(\mathfrak{M}_1, \mathfrak{M}, \mathfrak{M}_2)$ be a standard triple, X_0 be a bounded $\mathfrak{M}$ manifold and X be a semicomplete $\mathfrak{M}_1^1$ manifold satisfying a mild technical condition (the existence of $\mathfrak{M}_1$ exponential maps) then the set of $\mathfrak{M}$ maps, $\mathfrak{M}(X_0, X)$, is a semicomplete $\mathfrak{M}_2$ manifold in a natural way. If $F: X \to X_1$ is an $\mathfrak{M}_1$ map of semicomplete $\mathfrak{M}_1^1$ manifolds admitting $\mathfrak{M}_1$ exponentials then $F_*: \mathfrak{M}(X_0, X) \to \mathfrak{M}(X_0, X_1)$ is an $\mathfrak{M}_2$ map. If $h: X_0 \to X_2$ is an $\mathfrak{M}$ map of bounded $\mathfrak{M}$ manifolds then $h^*: \mathfrak{M}(X_2, X) \to \mathfrak{M}(X_0, X)$ is an $\mathfrak{M}_2$ map. If X_1 is a semicomplete $\mathfrak{M}_1^2$ manifold admitting $\mathfrak{M}_1^1$ exponentials then, since $(\mathfrak{M}_1^1, \mathfrak{M}, \mathfrak{M}_2^1)$ is a standard triple, $\mathfrak{M}(X_0, X)$ is an $\mathfrak{M}_2^1$ manifold and its tangent bundle can be naturally identified with $\tau_{X*}: \mathfrak{M}(X_0, TX) \to \mathfrak{M}(X_0, X)$. Under this identification $T(F_*)$ is identified with $(TF)_*$ and $T(h^*)$ is identified with h^*.

On composition as a function of two variables, the following result is typical: Let X_1 be a semicompact $\mathcal{L}^{s+t+1}$ manifold, X_2 be a semicomplete $\mathcal{L}^{s+r+t+2}$ manifold (both admitting suitable exponential maps) and X_0 be a bounded $\mathcal{C}^t$ manifold $(r \geq s + 1)$. Let G be open and bounded in $\mathcal{C}^t(X_0, X_1)$. $\Omega_G(f) = f_*|G$ defines $\Omega_G: \mathcal{L}^{s+t}(X_1, X_2) \to \mathcal{L}^s(G, \mathcal{C}^t(X_0, X_2))$ an

$\mathcal{L}^{r-1}$ map. The composition map Comp: $\mathcal{C}^t(X_0,X_1) \times \mathcal{L}^{s+t}(X_1,X_2) \to \mathcal{C}^t(X_0,X_2)$ is an $\mathcal{L}^s$ map. Note that smoothness of Ω_G doesn't make sense until manifolds of maps are defined with noncompact domains. Ω_G is better behaved than Comp in that its smoothness increases with r, i.e. with the smoothness of X_2, while that of Comp does not.

While the definitions of metric structures on manifolds require a standard function space type, one can usually globalize nonstandard function space types on sufficiently smooth semicomplete manifolds. For example, on manifolds of finite type (see Chapter VIII) the Sobolev function space types can be globalized essentially the same way as on compact manifolds (eg. [30; Section 25]). While we don't consider the Sobolev spaces further in this work, Chapter V carries out this globalization for $\mathcal{C}_u^r$ and $\mathcal{L}_a^r$ $(0 < a \leq 1)$, the derived function space types for uniformly continuous and Hölder continuous functions. $\mathcal{C}_u^r$ and $\mathcal{L}_a^r$ bundles, sections and maps can be defined on semicomplete $\mathcal{L}^r$ manifolds. Actually for these special function space types the globalization is carried out in a more general context.

In various applications a manifold carries an auxiliary topology coarser than the manifold topology. For example, in looking at compact hyperbolic invariant subsets for a dynamical system, it is technically useful to think of the subset as a discrete space (and hence a semicompact manifold) and regard the original topology as such an auxiliary topology. Again, given a foliation of a compact manifold it is useful to consider the leaf space as a nonseparable, semicompact manifold and regard the original topology as auxiliary. These are both special cases of leaf immersions, examined in Chapter VII. In Chapter V auxiliary pseudometrics on a regular metric manifold (apm's) are defined as pseudometrics coarser than the admissible metrics. Structures are defined with the r-jet of everything uniformly continuous, or Hölder continuous with respect to the apm.

Chapter VI contains a description of immersions, submersions and transversality in the metric category. If X_0 and X_1 are regular $\mathfrak{M}^1$ metric manifolds $f: X_0 \to X_1$ is an <u>$\mathfrak{M}^1$ immersion</u> if $\rho_0 > f^*\rho_1$ and there exist admissible atlases $\mathfrak{a}_0 = \{U_\alpha, h_\alpha\}$ and $\mathfrak{a}_1 = \{V_\beta, g_\beta\}$ on X_0 and X_1 such that $f: \mathfrak{a}_0 \to \mathfrak{a}_1$ is index preserving and for each α, the principal parts of f, $g_{\beta(\alpha)} \cdot f \cdot h_\alpha^{-1}$ are restrictions of inclusions of a factor into a product of Banach spaces. f is then an $\mathfrak{M}^1$ map and $T_x f: T_x X_0 \to T_{fx} X_1$ is a split injection, i.e. f is an immersion in the usual sense. If $j: E_0 \to E_1$ is a split injection of Banach spaces we define the splitting constant $\theta(j) = \max(\|j\|, \inf\{\|P\|: P: E_1 \to E_0 \text{ with } P \cdot j = I\})$. If $\| \ \|_i$ is a Finsler on τ_{X_i} associated with the metric structure (such Finslers are defined in Chapter III and are called admissible Finslers) then $\| \ \|_0$ and $\| \ \|_1$ make $T_x X_0$ and $T_{fx} X_1$ into Banach spaces and so we can define $\theta(f)(x) = \theta(T_x f)$. If f is an $\mathfrak{M}^1$ immersion then $\rho_0 > \theta(f)$. Conversely, if f is an $\mathfrak{M}^1$ map which satisfies $\rho_0 \sim f^*\rho_1$ (such a map is called metricly proper) and f is an immersion with $\rho_0 > \theta(f)$ then f is an $\mathfrak{M}^1$ immersion. For submersions and transversality the situation is similar. In each case an atlas definition is the appropriate one but tests are developed using the classical definition and some global condition like domination of a splitting bound like $\theta(f)$. Thus, all of these notions are global ones in the metric category rather than local as in the differentiable category. The global conditions easy enough to manipulate that, for example, openness of the proper metric notion of transversality still holds. However, the density theorems of transversality theory are lost.

While we prove that every paracompact C^r manifold admits semicompact $\mathcal{C}^r$ structures (Chapter VIII, Section 4), this result is mainly of negative interest. I suspected the existence of finite dimensional, connected manifolds which did not admit semicompact structures. Such a manifold could not occur as a leaf of foliation of a compact manifold, answering a question of Sondow [26]. In applying the metric theory to a problem, the mere existence of semicompact structures is not too useful because choosing a metric structure and going to the metric category restricts the maps, vectorfields, etc.with which one can deal to those which are metric with respect to the chosen structure. Thus, it is of more interest to look for metric structures naturally associated with the problem. This is a matter of looking for associated atlases $\mathfrak{a}$ and estimating $\rho_{\mathfrak{a}}$ and $\lambda_{\mathfrak{a}}$.

For example, if X is a Lie group then left translates (or right translates) of a chart about the identity form an atlas generating a semicompact structure called the left semicompact structure (resp. the right semicompact structure). Left invariant (resp. right invariant) gadgets are metric with respect to this structure. The left and right structures are usually not the same but are contained in a semicomplete structure with admissible bound $\rho(x) = \max(\|Ad(x)\|, \|Ad(x)^{-1}\|)$ where Ad is the adjoint representation and the norm is computed using any fixed norm on the Lie algebra.

If X is any Riemannian manifold then the natural atlas is the atlas of normal coordinates indexed by the points of X. $\rho_{\mathfrak{a}}^{\mathcal{C}^r}$ of this atlas can be estimated using the Jacobi equation by dominating the norm of the curvature tensor and r - 1 of its covariant derivatives. $\lambda_{\mathfrak{a}}(x)$ is essentially the distance to the cut locus.

Lack of space has prevented the inclusion of the proofs of the above remarks in this work. I hope to deal with the relations between the metric theory and differential geometry elsewhere. However, in Chapter VII we discuss Grassmanians and apply them to the following type of question: Let X_0 be a semicomplete $\mathfrak{M}^r$ manifold ($r \geq 1$) and $f: X \to X_0$ be a C^r immersion. When does X carry an $\mathfrak{M}^r$ structure such that f is an $\mathfrak{M}^r$ immersion? The Grassmanian $G(TX_0)$ and the associated lifting

$G(f): X \to G(TX_0)$ allow us to construct natural atlases on X associated with f which show that the $\mathfrak{M}^r$ structure, if it exists, is unique. The existence problem is handled inductively using smoothing results like the following: Let $f: X \to X_0$ be an $\mathfrak{M}^r$ immersion $(r \geq 1)$ and X_0 be an $\mathfrak{M}^{r+1}$ manifold. There exists an $\mathfrak{M}^{r+1}$ structure on X with respect to which f is an $\mathfrak{M}^{r+1}$ immersion iff the $\mathfrak{M}^{r-1}$ map $G(f): X \to G(TX_0)$ of $\mathfrak{M}^r$ manifolds is in fact an $\mathfrak{M}^r$ map.

In Chapter VII, we also consider leaf immersions. Let X_0 be a semicomplete $\mathcal{L}^r$ manifold with d_0 some admissible atlas metric. Let X be a semicomplete $\mathcal{C}^r$ manifold and $f: X \to X_0$ be a $\mathcal{C}^r$ immersion. f is called a $\mathcal{C}^r_u$ leaf immersion if f is a $\mathcal{C}^r_u$ map with respect to the apm $f^*d_0 = d_0 \cdot f \times f$ on X (in the sense of Chapter V). Inductively define $G^0(X_0) = X_0$, $G^0(f) = f$ and $G^{i+1}(X_0) = G(TG^i(X_0))$, $G^{i+1}(f) = G(G^i(f)): X \to G^{i+1}(X_0)$ and choose d_i some admissible atlas metric on $G^i(X_0)$. A $\mathcal{C}^r$ immersion f is a $\mathcal{C}^r_u$ leaf immersion iff f^*d_0 is uniformly equivalent to $G^r(f)^*d_r$. The standard example of a leaf immersion is the "inclusion" of the leaf space of a foliation into the ambient manifold. Leaf immersions were introduced by Hirsch, Pugh and Shub [12] who proved the existence of an atlas on the domain, called a plaquation atlas, which resembles a foliation atlas in some respects. We prove that a bijective $\mathcal{C}^r_u$ leaf immersion with closed image is a lamination or partial foliation in the sense of Ruelle and Sullivan [25] iff a certain weak factoring property holds.

A semicomplete, finite dimensional manifold X is said to be of <u>finite type</u> if there exist an s admissible atlas $\mathcal{A} = \{U_\alpha, h_\alpha\}$ and an integer N such that for each $x \in X$, $x \in U_\alpha$ for at most N values of α, i.e. if the nerve of the cover $\{U_\alpha\}$ has dimension $< N$. A classical result of dimension theory assures that with $N = \dim X + 1$ admissible atlases exist with this property. However, I was unable to preserve the uniformity of the open cover in getting such an atlas and so have been unable to prove the obvious conjecture that every semicomplete, finite dimensional manifold is of finite type. However, a rich stock of manifolds of finite type is provided by the theorem that if X is a semicomplete manifold which $\mathcal{L}^1$ immerses in X_0, a manifold of finite type, then X is of finite type. Conversely, any $\mathfrak{M}^r$ manifold of finite type $\mathfrak{M}^r$ immerses in some Euclidean space. This result and many other translations into the metric category of standard results of differential topology follow for manifolds of finite type because for such manifolds partitions of unity are available in the metric category. Using partitions of unity and transversality theory we are able to prove standard smoothing results in the metric category for manifolds of finite type. In particular, any $\mathfrak{M}^r$ metric structure of finite type can be smoothed to obtain a $\mathcal{C}^\infty$ structure of finite type, i.e. there exist in the $\mathfrak{M}^r$ structure adapted atlases $(\mathfrak{a},\rho)$ with $\rho > 1/\lambda_{\mathfrak{a}}$, $\rho > \rho^{\mathcal{C}^t}$ for $t = 1,2,\dots$, and the nerve of $\mathfrak{a}$ is finite dimensional.

Notation: All pseudometrics in this work are allowed to take the value ∞. $B^d(x,r)$ (or $B^d[x,r]$) is the open (resp. closed) d-ball about x with radius r. In cross references, we will drop the self-referrent part of a theorem's designation. Thus, Proposition III.7.2 occurs in Section 7 of Chapter III and in Chapter III it will be called Proposition 7.2 except in Section 7 where it will be called Proposition 2.

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