PRECALCULUS

PROBLEM SOLVING WITH TECHNOLOGY

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PREFACE

Technology affects almost every aspect of our lives. There has never been a greater need for a population with good mathematical skills and proper training in applying mathematics to solving problems. Technology demands people who can handle new challenges and apply their training to tackle problems never before encountered.

Fortunately, the same technology provides new tools and new ways to help us learn the mathematical skills and ideas needed. ***Precalculus: Problem Solving with Technology*** is a partial response to the changing demands on our students.

The whole area of introductory mathematics has been in ferment in the last decades of this century. There has been a profound revolution in mathematics education, with major national efforts in calculus reform, the adoption of standards for teaching and assessment of public school mathematics, and attempts at better articulation of college mathematics with the disciplines that depend on calculus. One significant driving force of this effort has been technology. With more emphasis on technology has come a re-examination of fundamental principles of pedagogy. Both pedagogical concerns and technology have significantly influenced the design and writing of this book.

Our assumption is that every student has consistent and convenient access to graphing or computational technology. A graphing calculator will certainly be the primary tool for most users, but the book can be used just as well with more sophisticated computer systems. We expect our students to learn to use graphing tools and to use them to explore their own mathematics. We expect active learners. Mathematics has always required student involvement, and this text provides guidance to make that involvement more productive, whether done individually, in small learning groups, or in full classroom discussions.

Our philosophy on the impact of technology on mathematics is summed up in the following two statements.

Some mathematics becomes more important because technology requires it;
some mathematics becomes less important because technology replaces it;
some mathematics becomes possible because technology allows it.

Technology provides powerful tools, but it has unavoidable limitations.

We strongly believe that technology should affect the precalculus curriculum. Many books that attempt to make use of graphing technology still cover every traditional topic, many in the old traditional ways, making technology just an adjunct that permits the treatment of more difficult problems.

Our attitude is that students should learn some topics in entirely new ways. Other topics should receive less emphasis (or be dropped entirely) because they no longer carry the same importance in a world where students have ready access to computer power. Graphing calculators are really hand-held computers, each with more computing power than entire universities had not many years ago.

We realize that not all students have access to the same technology. It is handy, but frequently impossible, for the teacher to have the same calculator or computer for all students. Even in schools where there are classroom sets of a single calculator, one classroom may have Texas Instrument machines and another may be supplied by Hewlett-Packard. We make a strong effort to be "calculator neutral." Our language, illustrations, and instructions are generic, not mimicking any specific calculator. To provide some help to students, however, we include an on-page pedagogical feature called Technology Tips, suggestions about using specific calculators for specific tasks.

We have no intention of making our students calculator experts. They will end up knowing much more than we could ever hope to teach them, anyway. Our aim is limited: we want each student to understand a few tasks well enough to make the calculator a tool for mathematical exploration, a means of visualizing mathematical concepts. Our Technology Tips address most of the general questions students will have about the kinds of calculator skills required, as well as opening up a number of unexpected vistas, ideas not found anywhere else in our experience.

An essential part of learning about calculators is an understanding of some of the limitations of computing technology, so most sections in which the calculator is discussed also include examples to suggest where the calculator cannot take us.

We endorse the principles enunciated in "For Good Measure," the report of the National Summit on Mathematics Assessment, and our exercises and examples specifically address each of the following from that report:

Encourage students to explore

Many exercises are titled *Explore,* directing students toward specific goals while inviting them to delve into open-ended explorations.

Help students to verbalize their mathematical ideas

Students are invited to verbalize appropriate strategies for a particular exercise or to explain why some result might be counterintuitive.

Show students that many mathematical questions have more than one right answer

Exercises titled *Your Choice* ask students to create their own examples satisfying given conditions, testing an understanding of basic underlying concepts.

Teach students, through experience, the importance of careful reasoning and disciplined understanding

Examples, many with strategies outlining the reasoning to be followed, step students carefully through numerous applied problems, followed by discussions of pitfalls or common errors, sometimes looking at the nature of solutions and how meaningful a particular result may be.

Provide evidence that mathematics is alive and exciting

In addition to casting examples in terms of contemporary and interesting applications, we include frequent Historical Notes humanizing some significant mathematical developments, as well as marginal quotations from contemporary mathematicians that suggest how these individuals came to discover mathematics or how they dealt with challenges on their way to their present stature. Capsule biographies appear in an appendix, "How They Came to Mathematics."

Content Highlights

Functions and Graphs Graphing technology can make functional behavior come to life for students. Terminology such as "increasing" or "decreasing" becomes clear in the context of pictures. Students learn functional notation naturally as they explore translations and dilations of graphs and see for themselves the difference between the graphs of $f(x) + c$ and $f(x + c)$. The language of functions is introduced in Chapter 2 and is a unifying theme throughout the text.

Roots and Zeros of Functions The solutions of both equations and inequalities have vivid meaning in terms of graphs. Students find that much of the mystery that has traditionally attended work with absolute values evaporates when they can draw their own graphs, and the algebraic manipulations required make more sense. (See Sections 1.4 and 1.5.) Students see new connections between algebraic and graphical information and find reinforcement with each new kind of function studied.

Polynomial and Rational Functions Some of the powerful theorems that have been developed for finding zeros of polynomial functions take on new meaning in the light of graphing technology. As an example, the rational zeros theorem is not needed as a means of starting the search for zeros of an integer polynomial function, but it yields information about the nature of zeros that the calculator cannot provide. On the other hand, calculators give us excellent decimal approximations beyond reach without technology. Relations between factored and expanded forms have geometric meaning. In exploring asymptotic behavior of rational functions, students learn how important different windows can be. The calculator can alert us to significant aspects of functional behavior, but if we fail to understand the underlying mathematical meaning, the calculator can just as easily hide important, subtle items. When students are asked to construct their own examples of polynomial functions with certain properties of turning points, they see if they really understand polynomial structure.

Parametric Equations Parametric equations allow us to view many topics in a new light. Parametric graphing is introduced in Chapter 2 and is used consistently throughout the text. Inverse functions are most natural when we interchange variables and see how the graph is reflected. Thus relations between the graphs of exponential and logarithmic functions, or graphs of trigonometric and inverse trigonometric functions are more easily seen. Circles and other conic section graphs are easy to produce. After an introduction to matrices (Section 9.6), we get matrix transformations to rotate axes, and with parametric graphing students can graph rotated conics (Section 10.4).

Exponential and Logarithmic Functions After a foundation of transformations of functions (Chapter 2), the essential unity of all exponential functions becomes transparent; any exponential function can be considered as a transformation of any other, and the natural exponential function is our choice as prototype. The same relations apply just as naturally to the function inverses, the logarithmic functions.

Trigonometric Functions The function concept and the dynamic possibilities of graphing provide more unity than has previously been available for introducing the trigonometric functions and their inverses. Graphing technology allows new approaches to identities, but limitations of the technology illustrate for students why there is something to prove.

Optimization and Problem Solving A consistent theme of the text is the proper formulation and solution of applied problems. We examine examples of problems leading to polynomial, exponential, trigonometric, linear, and nonlinear models. We discuss differences between exact and approximate solutions and consider where each may be appropriate or necessary. Students are constantly encouraged to check their results to see if they are reasonable or appropriate, leading naturally to ideas of estimation.

Exercises and Pedagogy Exercises are numerous and varied. Mathematical skills are developed and reinforced. Because we assume that every student uses a graphing calculator regularly, we do not segregate exercises that call on calculators, but most sets of related exercises are identified with capsule titles. Many exercises call for both algebraic and graphical solutions. Each exercise set begins with *Check Your Understanding* exercises, going beyond typical skills-based problems. They allow individuals to test understanding of key ideas, or they can be used for class discussion purposes. *Test Your Understanding* sets at the end of each chapter serve the same ends, pulling together the concepts of the whole chapter, followed by *Review for Mastery* exercises. A frequent feature is *Looking Ahead to Calculus*. These examples and exercises develop the algebraic and technological skills needed for certain kinds of problems normally encountered in a calculus course. Some exercises are labeled *Explore* or *Your Choice,* which may be used individually as enrichment or as a basis for group or written projects. Many exercises ask students to verbalize their responses, to explain or relate ideas, and can be used for varied purposes by the instructor.

Technology Tips All graphing calculators that are now widely available will do many more things than we ask of our students in this text. Several calculators have built-in routines to approximate solutions to transcendental equations or find complex zeros of polynomials or solve systems of linear equations; some have split-screen graphing or tabular function displays. We deal essentially only with calculator capabilities that are common to the following calculators:

TI-81, TI-82, TI-85,
HP 48G and 48GX, and the HP 38G,
Casio fx-7700 (or 9700 or 9900).

Our Technology Tips help students to use all of these calculators to do specific tasks required in the text. Instructors, of course, have the option of teaching their students to use other calculator or computer routines that may be available, and they can make appropriate adjustments in assigning exercises or designing test questions.

This text is based in part on the second edition of our *Precalculus* but it has been rewritten from the ground up. We have re-examined every topic, every section, and every exercise. Many of the changes are those suggested by our work with pilot sections of classes in which all students have graphing calculators. In each such class we get confirmation of the fact that our students have lots to teach us, their teachers, in addition to what they teach each other if we give them an opportunity.

Supplement Package to Accompany the Text

This text is accompanied by the following supplements:

For the Instructor. The *Instructor's Guide* includes solutions to all of the text exercises. A collection of problems for each chapter can be used for tests.

The *HarperCollins Test Generator for Mathematics* is one of the top testing programs on the market for IBM and Macintosh computers. It enables instructors to select questions for any section in the text or to use a ready-made test for each chapter. Instructors may generate tests in multiple-choice or open-response formats, scramble the order of questions while printing, and produce up to 25 versions of each test. The system features printed graphics and accurate mathematical symbols. The program also allows instructors to choose problems randomly from a section or problem type, or to choose problems manually while viewing them on the screen with the option to regenerate variables. The editing feature allows instructors to customize the chapter disks by adding their own problems. This is especially important in designing questions that are appropriate for students.

The *QuizMaster On-Line Testing System,* available in both IBM and Macintosh formats, coordinates with the HarperCollins Test Generator and allows instructors to create tests for students to take at the computer. The test results are stored on disk so the instructor can view or print test results for a student, a class section, or an entire course.

For the Student. The *Student's Solution Manual* contains detailed solutions to the odd-numbered exercises. It is available for student purchase; ask your college bookstore manager to order ISBN 0-673-99971-8.

Precalculus Investigations Using DERIVE (0-673-99097-4) by David Mathews of Longwood College and *Precalculus Investigations Using MAPLE* (0-673-99410-4) by David Mathews and Keith Schwingendorf, of Purdue University-North Central, will help you to integrate technology into your precalculus course. Twelve lab exercises provide carefully structured, interactive learning environments for students. The manual includes real-world applications, concept overviews, and lab reports.

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Lawrence O. Cannon
Joseph Elich

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HISTORICAL NOTES

1 BASIC CONCEPTS: REVIEW AND PREVIEW

In CHAPTER 1 WE CONSIDER the nature of mathematics, where mathematics comes from, and how it is used. This chapter lays a foundation for the entire book. Section 1.1 describes how mathematical models represent real-world problems, including calculator use and approximations. Sections 1.2 and 1.3 review terminology and the properties of numbers related to ordering and absolute values. Section 1.4 introduces the ideas of graphs and their uses, both on a number line and in the plane. Section 1.5 reviews some of the techniques from elementary algebra, how these techniques relate to graphing technology, and how they allow us to find solution sets for a variety of kinds of open sentences. The final section demonstrates how to approach and formulate a number of different problems, introducing techniques that are useful throughout the rest of the book and all of the study of mathematics.

1.1 MATHEMATICS MODELS THE WORLD

> Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.
>
> *Joseph Fourier*

What Is Mathematics?

Consider these situations and note what they have in common:

1. At the edge of the Beaufort Sea, north of the Arctic Circle, a dozen adults of the Inuit people are tossing a young boy aloft on a human-powered trampoline made of a blanket.

2. From the observation deck of the Sears Tower (the world's tallest building at 1454 feet) a visitor can see nearly six miles further out into Lake Michigan than someone at the top of the John Hancock Center (1127 feet tall).
3. A pilot of a Goodyear blimp heading south over Lake Okeechobee at 5300 feet wants to estimate the time remaining before visual contact with the Orange Bowl, where a football game is to be televised.

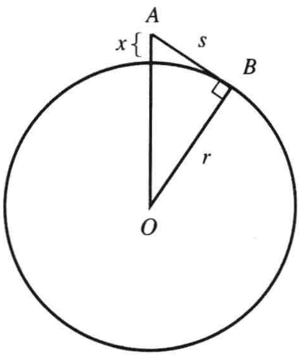

FIGURE 1
How far can you see from x miles above the earth?

Each of these situations deals with the curvature of the earth's surface and the fact that it is possible to see farther from a higher elevation. The Inuits want to get an observer high enough to see whether the pack ice is breaking up in the spring; the Sears Tower is 327 feet taller than the Hancock Center; at an elevation just over a mile, how far can the blimp pilot see?

Mathematics strips away the differences in these situations and finds one simple model to describe common key features. Figure 1 shows a cross section of the earth as a circle with center at O and radius r. Our model assumes a spherical earth, a fairly good approximation of the truth. From point A located x feet above the earth, the line of sight extends to B. (Line AB is tangent to the circle and hence perpendicular to radius OB.)

All of the situations listed above fit in this structure. The Inuit boy, the visitor to the top of a skyscraper, and the blimp pilot could each be seen as located at point A for different values of x. For any given x, applying the Pythagorean theorem (discussed in Section 1.4) to the right triangle AOB gives the corresponding distance s to the horizon.

$$r^2 + s^2 = (r + x)^2, \text{ where } x, r, \text{ and } s \text{ are in miles.}$$
$$r^2 + s^2 = r^2 + 2rx + x^2$$
$$s^2 = 2rx + x^2$$
$$s = \sqrt{2rx + x^2}.$$

Part of the power of mathematics comes from its capacity to express in a single sentence truths about several seemingly diverse situations. The solution to one equation automatically applies to any other application that gives rise to the same equation. The expression for s can be used to solve any of the problems listed above. See Exercises 39–44.

Geometry really turned me on. My father taught me by giving me problems to solve. He gave me thousands of geometry problems while I was still in high school. After he gave me one and I came back with a solution, he would say, "Well, I'll give you another one." The solving of thousands of problems during my high school days—at the time when my brain was growing—did more than anything else to develop my analytic power.
George Dantzig

Mathematics and the Real World

Much of the importance and vitality of mathematics comes from its relationship with the world around us. Humans invented numbers to count our sheep; we created rules for addition and multiplication as we needed to barter or compare land holdings. As human understanding of the world grew more sophisticated, mathematical tools grew as well. Sometimes mathematical curiosity led people in unexpected directions and their explorations became important for their own sake.

Mathematics is a lively part of our intellectual heritage. Some of the most intriguing and challenging mathematical investigations grew out of attempts to answer seemingly innocuous questions or understand simple observations. The most lasting and significant human achievements are direct consequences of our desire to understand and control the world.

Mathematics and Mathematical Models

When we encounter a problem whose solution involves the use of mathematics, we must decide how much detail is essential. In the line-of-sight examples the solution assumed the earth as a perfect sphere. The differences between that mathematician's earth and the actual globe are substantial. The equation for the distance to the horizon (s miles) implies that someone could see more than 40 miles from the top of either the Sears Tower or the Hancock Center; on a clear day someone in Chicago might want to check that conclusion.

In a mountain valley ringed by peaks that rise several thousand feet, it isn't possible to see 40 miles in any direction, even from 1500 feet up. Does that invalidate our mathematical model? Of course not; we must know something about particulars when we interpret a result. Questions about the way the world works frequently require simplifying assumptions to make the problems more tractable. See the Historical Note, Mathematical Models and Gravity (p. 135).

Technology

We assume that every student has access to some kind of graphing technology that permits graphing functions. Yours may be as simple as a graphing calculator or as complex as sophisticated computer software. *We use the language of graphing calculators* in this text, but you can use any available technology to do the work. If you are working with technology that is new to you, perhaps the most important thing is to experiment freely so that you become comfortable and confident with your own tools. Verify every computation in our examples. Talk with others about what works and what doesn't. Make sure that you can produce the same kinds of pictures that we show in the text.

Each graphing calculator and computer graphing software package is different; display screens have different proportions, and commands and syntax vary. We cannot give instructions to fit every kind of machine, but it should be possible to duplicate our computations and calculator graphs on almost any kind of graphing technology you have available. In our Technology Tips we make suggestions that may be helpful. If it seems that your calculator won't do something we are describing, discuss it with your instructor, look at your owner's manual, and ask classmates. There may be another way to get around the problem.

Calculators and computers have become incredibly powerful, but they remain limited. While they can do wonders, they may still properly be called "Smart-Stupids," a name coined by Douglas Hofstadter. However amazing their computing power, the machine is not smart enough to know that we *meant* to press [+] when we pressed the [÷] key.

Approximate Numbers and Significant Digits

When we use mathematics to model the real world, we have to realize that measurements of physical quantities can be only approximations. A biologist may be able to count exactly the number of eggs in a bird's nest, but comparing the volume or weight of two eggs requires *approximate numbers,* since any number we use is only as good as our measuring device. We also use approximate numbers when we need a decimal form for a number such as $\sqrt{3}$.

Questions involving approximations entail decisions about the tolerable degree of error. Error tolerance decisions usually hinge on concerns other than mathematics, but all of us must make such decisions in working problems that involve measurements or when we use calculators. We need guidelines.

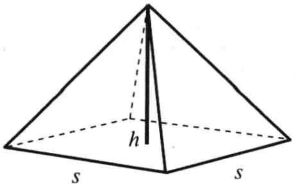

FIGURE 2
The volume of a pyramid of height h and sides s: $V = \frac{1}{3}s^2h$.

Perhaps the greatest problem in working with calculators is interpreting displayed results. When we enter data, the calculator returns so many digits so quickly and easily that we may think we have gained more information than we really have. This difficulty can be illustrated by an example from a recent calculus text. The book derives an equation for the volume of a pyramid, as shown in Figure 2, and then applies the formula to find the volume of the Great Pyramid of Cheops. The original dimensions are given (approximately) as

$$s = 754 \text{ feet} \quad \text{and} \quad h = 482 \text{ feet}.$$

When we substitute these numbers into the formula, a calculator immediately displays 91341570.67, from which the authors conclude that the volume is "approximately 91,341,571 cubic feet." In the following example, we illustrate why we are not justified in rounding to the nearest cubic foot, even if the values for s and h are measured to the nearest foot.

►EXAMPLE 1 ***Appropriate rounding*** Assuming that the height h and side length s are measured to the nearest foot, giving 482 feet for h and 754 for s, how much variation can this leave in the computed volume, using $V = \frac{s^2h}{3}$?

Strategy: Let V_0 be the volume using the smaller values of s and h, while V_1 is the volume using the larger values of s and h. Compute V_0 and V_1, and then compare the results.

Solution

To say the linear measurements are correct to the nearest foot means that they satisfy the inequalities

$$753.5 < s < 754.5 \quad \text{and} \quad 481.5 < h < 482.5.$$

Using the smaller values for s and h gives

$$V_0 = \frac{(753.5)^2(481.5)}{3} \approx 91{,}125{,}841.12.$$

The upper values for s and h yield

$$V_1 = \frac{(754.5)^2(482.5)}{3} \approx 91{,}557{,}631.87.$$

The difference between V_1 and V_0 is

$$V_1 - V_0 \approx 431{,}790.75.$$

The computed and actual volumes could differ by nearly *half a million cubic feet!* See Example 3. ◄

The world of mathematics is an *ideal* world, dealing with exact numbers and precise relationships, but mathematics also says much about the inexactitude and fuzziness of the physical world. In applying mathematics, we create a precise model to mirror an imprecise reality. Whenever mathematics delivers an answer for an applied problem, we must ask what the numbers mean and what degree of significance they have for the original problem.

What Do the Digits Mean?

What is the diagonal of a square that measures a mile on each side? The mathematical model of a square of side 1 has a diagonal of exactly $\sqrt{2}$. Our calculator displays 1.414213562 for $\sqrt{2}$. For a mile, what does each of these decimal places