# The Theory of Ruled Surfaces

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# OF
# RULED SURFACES

BY

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CAMBRIDGE
AT THE UNIVERSITY PRESS
1931

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Tokyo, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9781107689671



First published 1931
First paperback edition 2011

*A catalogue record for this publication is available from the British Library*

ISBN 978-1-107-68967-1 Paperback

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# PREFACE

In this volume all the ruled surfaces in ordinary space of orders up to and including the sixth are studied and classified. Tables shewing the different types of surfaces are given towards the end of the book, and the tables for the surfaces of the fifth and sixth orders are here obtained for the first time.

It seems that the results so obtained are of great importance; but the incidental purpose which, it is hoped, may be served by the book is perhaps of still greater importance. For there exists at present no work, easily accessible to English readers, which tests the application of the general ideas here employed in anything like the same detail. One might mention especially the use of higher space and the principle of correspondence, and these two ideas are vital and fundamental in all modern algebraic geometry. It is hoped therefore that the book may be of use to a wide circle of readers.

I wish to express here my thanks to the staff of the University Press for their unfailing accuracy in the printing and for the ready courtesy with which they have accepted my suggestions.

Notwithstanding the large number of surfaces which are herein investigated, the book would be incomplete were I not to make an acknowledgment of my obligations to Mr White, of St John's College, and Professor Baker. Even those who have only a slight knowledge of the multifariousness of Mr White's mathematical public services will be surprised to learn that he found time not only to read the proof sheets but also to read through the whole of the manuscript, and I am very grateful to him for his criticisms and suggestions.

My gratitude to Professor Baker is something more than that of a student to his teacher. He it was who first suggested that I should undertake this work, and his encouragement has been given unsparingly—and effectively—in times of difficulty. I have derived great benefit not only from my personal conversations with him but also from attending his courses of lectures. I thank him for many things; but especially for his interest, which has never flagged, and for his trust, which has never wavered.

W. L. E.

TRINITY COLLEGE
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*October* 1930

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# CHAPTER I

# INTRODUCTORY

## *SECTION I*

## PRELIMINARIES

**1.** The system of points on a line is determined by two of them, any third point of the line being derivable from these two; the same line is equally well determined by any two of its points. Similarly, if three points are taken which are not on the same line they determine a plane, the same plane being equally well determined by any three non-collinear points of it. Proceeding in this way we say that $n + 1$ independent points determine a linear space of $n$ dimensions, the points being independent when they are such that no one of them belongs to the space of less than $n$ dimensions determined by the others; the same space of $n$ dimensions is equally well determined by any $n + 1$ independent points belonging to it.

We shall use the symbols $[n]$ and $S_n$ to denote a space of $n$ dimensions. In $[n]$ two spaces $[m]$ and $[n - m]$ of complementary dimensions have, in general, one point in common and no more. A space $[p]$ and a space $[q]$ have, in general, no common points if $p + q < n$, while if $p + q > n$ they have, in general, a common $[p + q - n]$. If they have in common a space $[r]$ where $r > p + q - n$, then they are contained in a space $[p + q - r]$ or $[n - s]$, where $s = r - p - q + n$. For example: two lines in ordinary space do not intersect in general; if they do so they lie in a plane. If we call the intersection of $[p]$ and $[q]$ their *meet* and the space of lowest dimension which contains them both their *join*, then the sum of the dimensions of the meet and the join is $p + q$.

**2.** Just as we can project, in ordinary space, on to a plane so we can project, in $[n]$, on to $[n - 1]$; if $O$ is the centre of projection and $P$ any point of $[n]$ the line $OP$ meets $[n - 1]$ in a point $P_1$ which is the projection of $P$. We can then project again from a point $O_1$ of $[n - 1]$ on to a space $[n - 2]$ in $[n - 1]$, the line $O_1P_1$ meeting $[n - 2]$ in a point $P_2$. The passage from $P$ to $P_2$ can, however, be carried out in one step, simply by joining $P$ to the line $OO_1$ by a plane and taking $P_2$ as the intersection of the plane with $[n - 2]$. We thus speak of projecting the points of $[n]$ from a line on to $[n - 2]$. Similarly, we can project from a plane on to an $[n - 3]$, from a solid on to an $[n - 4]$, and so on; the sum of the dimensions of the space which is the centre of projection and of the space on to which we are projecting being always $n - 1$.

**3.** Just as the order* of a plane curve is the number of points in which it is met by a line, and the order of a twisted curve is the number of points in which it is met by a plane, so the order of a curve in [4] is the number of points in which it is met by a solid† and so on, the order of a curve in $[n]$ being the number of points in which it is met by a space $[n-1]$ of complementary dimension. The order of a surface in $[n]$ is the number of points in which it is met by a space $[n-2]$, just as the order of a surface in ordinary space is the number of points in which it is met by a line. It is here implied that the space $[n-2]$ has a general position in regard to the surface, otherwise it might meet it in a curve; a line in ordinary space may itself lie on a surface. Similarly the order of a locus of $r$ dimensions is equal to the (finite) number of points in which it is met by a space $[n-r]$ of complementary dimension and of general position. A locus of dimension $r$ and order $m$ will be denoted by a symbol $M_r^m$ or $V_r^m$, and if $r = n-1$ the locus will be spoken of as a "primal." A space $[n-1]$ lying in $[n]$ is called a "prime" of $[n]$.

**4.** If we have a curve in ordinary space its chords fill up the space; there is a finite number of them passing through a point of general position. But in [4] the chords of a curve do not fill up the space; they form a locus of three dimensions whose order is the number of points in which it meets a line. If we have a system of coordinates in [4], say five homogeneous or four non-homogeneous coordinates, the locus is given by an equation in these coordinates, and the order of the locus is the order of this equation. In $[n]$ the chords of a curve form a three-dimensional locus whose order is equal to the number of points in which it meets an $[n-3]$. The chords of a surface form a five-dimensional locus.

**5.** Suppose that we have a curve of order $N$ in $[n]$; there may be a point of the curve such that any $[n-1]$ passing through it only meets the curve in $N-2$ other points. Such a point is called a double point of the curve. In particular we have the double points of a plane curve; for example, the point $x = y = 0$ is a double point on the cubic curve

$$x^3 + y^3 = 3xyz,$$

any line through it meeting the curve in only one further point. It is known that a plane curve of order $N$ cannot possess more than

$$\tfrac{1}{2}(N-1)(N-2)$$

double points, a $k$-ple point (i.e. a point such that any line through it meets the curve only in $N-k$ further points) counting as $\frac{1}{2}k(k-1)$ double

* It is always to be understood that the curves and loci spoken of are *algebraical.*

† The word *solid* will always mean a three-dimensional space. We shall sometimes find it convenient to use the word solid as well as the symbols [3] and $S_3$.

points*. If $d$ is the actual number of double points possessed by the plane curve the number $\frac{1}{2}(N-1)(N-2)-d$ was called by Cayley the *deficiency* of the curve. This number is in fact the same as the *genus* of the curve. The most fundamental property of the genus is that it is invariant for birational transformation of the curve; the genus of a curve in space of any number of dimensions can therefore be defined as the deficiency of a plane curve with which it is birationally equivalent.

The explanation of what is meant by *birational transformation* must be given here. Two curves are said to be *birationally equivalent* or to be in *birational correspondence* when the coordinates of a point on either curve are rational functions of the coordinates of a point on the other. In this way to a given point of either curve there will correspond one and only one point of the other; but multiple points will prove exceptions to this rule, to a multiple point on one of the curves there will correspond several points on the other. Thus we can say that there is a (1, 1) correspondence between the two curves, with certain reservations as to the multiple points. But it appears that we can always regard a multiple point as consisting of several points on different *branches* of a curve, and if we regard the multiple point in this way we can say that the correspondence is (1, 1) without exception. Thus a birational correspondence and a (1, 1) correspondence between two curves mean the same thing; and the fundamental property of the genus is that it is the same for two curves which are in (1, 1) correspondence.

If we are considering correspondences between the points of two curves, or between the points of a single curve, then a double point must be regarded as two distinct points on different branches of the curve. At a cusp, however, there is only a single branch.

In the quadratic transformation

$$x=\frac{1}{X},\qquad y=\frac{1}{Y},\qquad z=\frac{1}{Z},$$

the rational quartic $$y^2z^2+z^2x^2+x^2y^2=0,$$

with nodes at the three vertices of the triangle of reference, is transformed into the conic

$$X^2+Y^2+Z^2=0,$$

and to each node of the quartic there correspond two distinct points of the conic. Corresponding to the node $y=z=0$ we have the two points in which the conic is met by the line $X=0$; and to either of these points on the conic corresponds the node $y=z=0$ on the quartic, the two points on the conic giving points on two distinct branches of the quartic.

* It may be equivalent to more than this number of double points if the $k$ tangents are not all distinct or are such that some of them meet the curve in more than $k+1$ (instead of exactly $k+1$) points at the multiple point.

On the other hand, the rational quartic

$$y^2z^2 + z^2x^2 + x^2y^2 = 2xyz\,(x + y + z)$$

has cusps at the three vertices of the triangle of reference, and is transformed by the same transformation into the conic

$$X^2 + Y^2 + Z^2 = 2\,(YZ + ZX + XY).$$

Then to each cusp of the quartic there corresponds only one point of the conic, e.g. to the cusp $y = z = 0$ corresponds the point in which the conic is touched by its tangent $X = 0$.

**6.** When two curves are in (1, 1) correspondence it is of course not necessary that they should belong to spaces of the same number of dimensions; either of them can belong to a space of any number of dimensions. The genus therefore of a curve in $[n]$ is simply the genus or deficiency of the projection of this curve from a space $[n - 3]$ on to a plane; the correspondence between the curve and its projection will be (1, 1) if the $[n - 3]$ is of general position. A curve has the same genus as any curve of which it is the projection.

For example, we may project the curve of intersection of two quadric surfaces in ordinary space on to a plane from a point $O$. If $O$ is of general position in regard to the curve there are two and only two of its chords* which pass through $O$; the projection is a plane quartic with two double points and therefore of genus 1. Hence the curve of intersection of two quadrics is also of genus 1.

Of the $\infty^3$ possible positions of $O$ there are four (not on the curve) for which an infinity of chords of the curve pass through $O$, these being the vertices of the four quadric cones which belong to the pencil of quadrics containing the curve. The projection from one of these points does not give a (1, 1) correspondence but a (2, 1) correspondence, and the genus of the curve is altered by such a projection.

A curve of genus zero is said to be a *rational curve* because the coordinates of any point on it can be expressed as rational functions of a parameter, and this parameter can be so chosen as to be a rational function of the coordinates of a point of the curve†. Thus to each point of the curve corresponds one and only one value of the parameter and to each value of the parameter corresponds one and only one point of the curve. A rational curve is birationally equivalent to a straight line and all rational curves are birationally equivalent to one another.

A curve of genus 1 is said to be an *elliptic curve*; but it is not true that all elliptic curves are birationally equivalent to one another. There is belonging to an elliptic curve an invariant called its *modulus*; and in order

* Salmon, *Geometry of Three Dimensions* (Dublin, 1914), vol. 1, pp. 355, 356.

† If we have expressed the coordinates of a point of a curve as rational functions of a parameter and this parameter is *not* a rational function of the coordinates, we can always find a second parameter which is; the second parameter is a rational function of the first and the coordinates are rational in terms of it. See Lüroth, *Math. Ann.* 9 (1875), 163.

that two elliptic curves should be birationally equivalent it is necessary and sufficient that they should have the same modulus.

A curve of genus 2 is said to be *hyperelliptic*, although not all hyperelliptic curves are of genus 2*.

**7.** When we project a curve $C$ of order $N$ in $[n]$ on to any lower space the order of the projected curve is also $N$ provided that the centre of projection does not meet $C$. If the centre of projection is a space $[r]$ the space on to which we are projecting is an $[n-r-1]$; an arbitrary $[n-r-2]$ in this space meets the projected curve in a number of points equal to its order, and this number is the same as the number of points in which the $[n-1]$ joining $[n-r-2]$ to $[r]$ meets $C$. If $C$ is met in $M$ points by the centre of projection the projected curve is of order $N-M$. If we project on to a plane from an $[n-3]$ which does not meet $C$ we know that we shall obtain a plane curve of order $N$ with $\frac{1}{2}(N-1)(N-2)-p$ double points, where $p$ is the genus of $C$. But the space $[n-2]$ which joins $[n-3]$ to any one of these double points must, unless it contains a double point of $C$ itself, contain two different points of $C$; so that we have a chord of $C$ meeting $[n-3]$. Conversely, any chord of $C$ which meets $[n-3]$ gives rise to a double point of the projected curve. Thus, if $\delta$ is the number of actual double points of $C$, there must be $\frac{1}{2}(N-1)(N-2)-p-\delta$ chords of $C$ meeting an $[n-3]$ of general position; so that *the chords of $C$ form a three-dimensional locus of order $\frac{1}{2}(N-1)(N-2)-p-\delta$.*

**8.** *Normal curves.* We now introduce the important concept of a *normal curve*†. A curve is said to be *normal* when it cannot be obtained by projection from a curve of the same order in space of higher dimension. It is clear that no curve can lie in a space of higher dimension than the order $n$ of the curve, for taking any $n+1$ points of the curve we determine thereby a space of dimension $n$ at most, which contains the curve since it meets it in a number of points greater than its order. For example: a curve of the second order always lies in a plane.

The coordinates of a point of a rational curve of order $n$ in $[m]$ can be expressed as rational functions of a parameter $\theta$. If the coordinates are homogeneous, and so $m+1$ in number, the coordinates of a point of the curve can be taken as polynomials in $\theta$. Further, $\theta$ can be so chosen that it is a rational function of the coordinates (§ 6) so that to any given value of $\theta$ there corresponds one and only one point of the curve. Then none of the $m+1$ polynomials can be of degree higher than $n$, for otherwise a prime $S_{m-1}$, which is given by a single linear equation in the

* A curve of genus 2 is the simplest example of a class of curves which are said to be *hyperelliptic*. We can have hyperelliptic curves of any genus; but *all* curves of genus 2 are necessarily hyperelliptic. See e.g. Severi, *Trattato di Geometria Algebrica*, I, 1, 159 (Bologna, 1926).

† See Severi, *ibid.* 110–111.

coordinates, would meet the curve in more than $n$ points; while one polynomial at least must actually be of degree $n$. Thus a rational curve of order $n$ cannot lie in a space of dimension greater than $n$, since we cannot have more than $n+1$ linearly independent polynomials of order $n$ in $\theta$.

On the other hand, *a rational curve of order n can always be regarded as the projection of a rational curve of order n in* $[n]$*. If the curve is in $[m]$ we can suppose the homogeneous coordinates $x_0, x_1, \ldots, x_m$ of any point on it to be linearly independent polynomials of order $n$ in a parameter $\theta$. We can then choose $n-m$ further polynomials of order $n$ in $\theta$ such that all the $n+1$ polynomials are linearly independent; we then take a curve in $[n]$, the homogeneous coordinates $x_0, x_1, \ldots, x_n$ of a point on it being proportional to these polynomials. The former curve can be regarded as lying in the $[m]$ whose equations are $x_{m+1} = x_{m+2} = \ldots = x_n = 0$ and is the projection of the normal curve from the $[n-m-1]$ whose equations are $x_0 = x_1 = \ldots = x_m = 0$.

We can, merely by means of a linear transformation of the coordinates, take the coordinates of a point on a rational normal curve of order $n$ to be

$$x_0 = \theta^n, \quad x_1 = \theta^{n-1}, \ldots, x_r = \theta^{n-r}, \ldots, x_{n-1} = \theta, \quad x_n = 1.$$

The expressions $(\theta^2, \theta, 1)$ for a point on a conic and $(\theta^3, \theta^2, \theta, 1)$ for a point on a twisted cubic are well known.

The curve is given uniquely by the equations

$$\frac{x_0}{x_1} = \frac{x_1}{x_2} = \ldots = \frac{x_r}{x_{r+1}} = \ldots = \frac{x_{n-1}}{x_n},$$

or

$$\left\| \begin{matrix} x_0 & x_1 \ldots & x_r & \ldots x_{n-1} \\ x_1 & x_2 \ldots & x_{r+1} & \ldots \quad x_n \end{matrix} \right\| = 0.$$

Incidentally we have the equations of $\frac{1}{2}n(n-1)$ quadric primals containing the curve; these are linearly independent and any other quadric primal containing the curve is in fact linearly dependent from these.

The chords of the curve form the three-dimensional locus given by

$$\left\| \begin{matrix} x_0 & x_1 \ldots & x_r & \ldots x_{n-2} \\ x_1 & x_2 \ldots & x_{r+1} & \ldots x_{n-1} \\ x_2 & x_3 \ldots & x_{r+2} & \ldots \quad x_n \end{matrix} \right\| = 0,$$

which is of order $\frac{1}{2}(n-1)(n-2)$†.

* Veronese, *Math. Ann.* 19 (1882), 208.

† The coordinates of a point on the three-dimensional locus of chords are of the form

$$(\theta^n + \lambda\phi^n, \quad \theta^{n-1} + \lambda\phi^{n-1}, \ldots, \theta + \lambda\phi, \quad 1 + \lambda),$$

and depend on the three parameters $\theta$, $\phi$, $\lambda$.

For the order of the system of equations given by the vanishing of the determinants of a matrix see Salmon, *Higher Algebra* (Dublin, 1885), Lesson 19.