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Geometric Realizations of Curvature

Imperial College Press

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by Miguel Brozos Vázquez, Peter B. Gilkey & Stana Nikcevic

Preface

Many questions in modern differential geometry can be phrased as questions of geometric realizability; one studies whether or not certain algebraic objects have corresponding geometric analogues. One must examine the relationship between the algebraic category and the geometric setting to investigate the geometric consequences resulting from the imposition of algebraic conditions on the curvature. The decomposition of certain spaces of curvature tensors under the appropriate structure groups is crucial and motivates many investigations. Although we will primarily focus on the curvature tensor, there are other tensors which arise naturally and which also play an important role in our study. As we will often work in the indefinite setting, the structure groups are in general non-compact. This imposes some minor technical difficulties. In this book, we have attempted to organize some of the results in the literature which fall into this genre; as the field is a vast one, we have not attempted an exhaustive account but have rather focused on only some of the results in order to be able to give a coherent account.

We begin in Chapter 1 by introducing some notation and stating the main results of the book. We also outline in some detail the main results of the book and relate various results to the whole. The remainder of the book consists for the most part in establishing the results given here. In Chapter 2, we turn our attention to representation theory and derive the main results we shall need. Chapter 2 is self-contained with the exception of the results of H. Weyl and others concerning invariance theory for the orthogonal and unitary groups in the positive definite setting; the corresponding results in the higher signature setting and in the para-complex setting are then derived from these results. In Chapter 3, we present some classic results from differential geometry.

In Chapter 4, we work in the real affine setting and in Chapter 5, we work in the (para)-complex affine setting. In Chapter 6 we perform a similar analysis for real Riemannian geometry and in Chapter 7 we study (para)-complex Riemannian geometry. To the greatest extent possible, we present results in the para-complex and in the complex settings in parallel. We present following Chapter 7 a list of the main notational conventions used throughout the book. Following this list, we have included a lengthy bibliography. The book concludes with an index.

Each chapter is divided into sections; the first section of a chapter provides an outline to the subsequent material in the chapter. Theorems, lemmas, corollaries, and so forth are labeled by section. Equations which are cited are labeled by section; equations which are not cited are not labeled.

To comply with stylistic requirements for this series, a few non-standard usages have been employed for which we are not responsible. To begin with, the bibliographic style will be unfamiliar to almost all mathematical readers. For example, [Brozos-Vázquez et al. (2009)] refers to work by Brozos-Vázquez, Gilkey, Kang, Nikčević, and Weingart. On the other hand, [Brozos-Vázquez et al. (2009a)] refers to work by Brozos-Vázquez, Gilkey, Nikčević, and Vázquez-Lorenzo. The words *"para-Hermitian (+) or pseudo-Hermitian (−)"* have been used rather than the customary *"para/pseudo-Hermitian"*. There are a few other similar instances which we hope will not disturb the reader unduely. Es lo que hay.

Much of this book reports on previous joint work with various authors. It is an honor and a privilege to acknowledge the contribution made by these colleagues: N. Blažić, N. Bokan, E. Calviño-Louzao, J. C. Díaz-Ramos, C. Dunn, B. Fiedler, E. García-Río, R. Ivanova, H. Kang, E. Merino, J.H. Park, E. Puffini, K. Sekigawa, U. Simon, G. Stanilov, I. Stavrov, Y. Tsankov, M. E. Vázquez-Abal, R. Vázquez-Lorenzo, V. Videv, G. Weingart, D. Westerman, T. Zhang, and R. Zivaljevic. In addition to pleasant professional collaborations, they have enriched the personal lives of the authors.

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This book is dedicated to Ana, to Ekaterina, to George, and to Susana.

P. Gilkey, S. Nikčević, and M. Brozos-Vázquez February 2012

Contents

Chapter 1

Introduction and Statement of Results

A central area of study in differential geometry is the examination of the relationship between purely algebraic properties of the Riemann curvature tensor and the underlying geometric properties of the manifold. Many authors have worked in this area in recent years. Nevertheless, many fundamental questions remain unanswered. When dealing with a geometric problem, it is frequently convenient to work first purely algebraically and pass later to the geometric setting. For this reason, many questions in differential geometry are often phrased as problems involving the geometric realization of curvature.

The decomposition of the appropriate space of tensors into irreducible modules under the action of the appropriate structure group is central to our investigation and we review the appropriate results in each section. Many of the results in the book, although they involve non-linear analysis, are closely tied to the representation theory of the appropriate group and the corresponding linear subspaces. In contrast, other results are non-linear in their very formulation since one is studying orbit spaces under the structure group; these need not be linear subspaces.

In the remainder of Chapter 1, we summarize briefly the main results of this book to put them into context for the reader. We shall discuss the basic curvature decomposition results leading to various geometric realization results in a number of geometric contexts. This ensures that the various relations between these theorems are clearly and concisely presented; further details are presented subsequently.

We now outline briefly the contents of Chapter 1. In Section 1.1, we present some basic notational conventions. In Section 1.2, we sketch some of the representation theory we shall need; Chapter 2 will be devoted to the proof of these results.

The results of Section 1.3 and of Section 1.4 will be established in Chapter 3 and in Chapter 4. In Section 1.3 we treat affine geometry. Theorem 1.3.1 gives the decomposition [Strichartz (1988)] of the space of generalized curvature operators as a general linear module. The dimension of these modules is given in Theorem 1.3.2. The decomposition of Theorem 1.3.1 motivates the associated geometric realization results discussed in Theorem 1.3.3. In Theorem 1.3.4, we establish a basic geometric realization result for the curvature and the covariant derivative of the curvature of an affine connection or, equivalently, a connection with vanishing torsion tensor. In Section 1.4, we study mixed structures; this is the geometry of an affine connection in the presence of an auxiliary non-degenerate inner product. The curvature decomposition [Bokan (1990)] is stated in Theorem 1.4.1, the dimensions of the relevant modules are given in Theorem 1.4.2, and the associated geometric realization result presented in Theorem 1.4.3.

The results of Section 1.5 will be proved in Chapter 5. We return to affine geometry to treat (para)-Kähler affine curvature tensors. To emphasize the similarities, we shall for the most part present the complex and the para-complex settings in parallel. We present the curvature decomposition as (para)-complex general linear modules in Theorem 1.5.1 and as unitary modules in Theorem 1.5.2 [Matzeu and Nikčević (1991)] and [Nikčević (1992)]. This leads to the geometric realization result given in Theorem 1.5.3. The dimensions of these modules are stated in Theorem 1.5.4.

The results of Section 1.6 and of Section 1.7 will be established in Chapter 6. Section 1.6 treats Riemannian geometry. The Fiedler generators [Fiedler (2003)] for the space of Riemannian algebraic curvature tensors are given in Theorem 1.6.1. The fundamental curvature decomposition [Singer and Thorpe (1969)] is given in Theorem 1.6.2, and an associated geometric realization theorem by metrics of constant scalar curvature is presented in Theorem 1.6.3. In Section 1.7, we study Weyl geometry; this is midway in a certain sense between affine and Riemannian geometry. The extra curvature symmetry of Weyl geometry is given in Theorem 1.7.1, the curvature decomposition as an orthogonal module is given in Theorem 1.7.2, and the basic geometric realization result is given in Theorem 1.7.3. Theorem 1.7.4 gives various characterizations of trivial Weyl structures.

The results of Section 1.8, of Section 1.9, of Section 1.10, of Section 1.11, and of Section 1.12 will be established in Chapter 7. In Section 1.8, we turn our attention to (para)-complex geometry. The curvature decomposition

[Tricerri and Vanhecke (1981)] of the space of Riemann curvature tensors in the pseudo-Hermitian and in the para-Hermitian settings is given in Theorem 1.8.1. A geometric realization theorem is then presented in this context in Theorem 1.8.3. The dimensions of the associated modules are given in Theorem 1.8.2. If the almost (para)-complex structures $J_{\pm}$ are integrable, then there is an extra curvature condition [Gray (1976)]; we shall discuss this further in Theorem 1.9.1 in Section 1.9. The relevant geometric realizability results are outlined in Theorem 1.9.2 and rely on the curvature decompositions given previously. Theorem 1.9.3 is an algebraic fact related to these conditions.

(Para)-Kähler geometry is treated in Section 1.10. The (para)-Kähler curvature condition is given in Theorem 1.10.1 and the associated geometric realizability results are presented in Theorem 1.10.2. Additional curvature decomposition results are given in Theorem 1.10.3. In Section 1.11, we discuss Weyl geometry in the Kähler setting either for a complex or for a para-complex structure. We also discuss an analogous algebraic condition giving rise to curvature Kähler–Weyl geometry. We shall restrict our attention to dimensions $m \geq 6$ as the situation in dimension $m = 4$ is quite different. In Theorem 1.11.1, we show any Weyl structure which is (para)-Kähler is trivial and in Theorem 1.11.2, we give a similar characterization solely in terms of curvature. Theorem 1.11.4 is a similar result at the purely algebraic level. Theorem 1.11.3 generalizes Theorem 1.7.2 and Theorem 1.8.1 to Weyl geometry in the (para)-complex setting.

In Section 1.12, we change focus. Let $\nabla\Omega$ be the covariant derivative of the (para)-Kähler form. This has certain universal symmetries. In Theorem 1.12.1 we show that if H is any 3-tensor with these symmetries, then H is geometrically realizable as the covariant derivative of the (para)-Kähler form of some almost para-Hermitian manifold or of some almost pseudo-Hermitian manifold. This is based on an appropriate decomposition result (see Theorem 1.12.3); the relevant dimensions of the irreducible modules involved are given in Theorem 1.12.4.

Finally, in Section 1.13, we give a brief summary of results contained in [De Smedt (1994)] concerning hyper-Hermitian geometry for the sake of completeness.

It is worth giving a bit of an explanation about what we mean by geometric realizability since this is the focus of the book. Let $\{T_1, ..., T_k\}$ be a family of tensors on a vector space V. The structure $(V, T_1, ..., T_k)$ is said to be *geometrically realizable* if there exists a manifold M, if there exists a point P of M, and if there exists an isomorphism $\phi : V \to T_PM$ such that $\phi^* L_i(P) = T_i$ where $\{L_1, ..., L_k\}$ is a corresponding geometric family of tensor fields on M. Thus, for example, if $k = 1$ and if $T_1 = \langle \cdot, \cdot \rangle$ is a non-degenerate inner product on V, then a geometric realization of $(V, \langle \cdot, \cdot \rangle)$ is a pseudo-Riemannian manifold (M, g), a point P of M, and an isomorphism $\phi : V \to T_PM$ so that $\phi^* g_P = \langle \cdot, \cdot \rangle$.

1.1 Notational Conventions

In addition to the notation introduced here, more notation will be introduced subsequently as needed; a summary of the common notational conventions used in this book is to be found at the end just before the bibliography. Let M be a smooth manifold of dimension $m \geq 4$; there are often similar results in dimensions $m = 2$ and $m = 3$ that we will sketch in passing. Let V be a real vector space of dimension m. Let V^* be the associated *dual vector space.* We shall let $\{e_i\}$ be a basis for V and we shall let $\{e^i\}$ be the associated dual basis for V^*; when we wish to consider orthogonal bases, we will make this explicit. Setting $x^i := e^i(\cdot)$ defines coordinates $(x^1, \ldots, x^m)$ on V. Let $\partial_{x_i} := \frac{\partial}{\partial x_i}$. Adopt the *Einstein convention* and sum over repeated indices. We say that $x = c^i \partial_{x_i}$ is a *coordinate vector field* if the coefficients c^i are constant; this notion is independent of the particular basis chosen for V. If $\theta^2 \in \otimes^2 V^*$ and if $\theta^4 \in \otimes^4 V^*$, we expand

$$\theta^2 = \theta^2_{ij} e^i \otimes e^j \quad \text{and} \quad \theta^4 = \theta^4_{ijkl} e^i \otimes e^j \otimes e^k \otimes e^l$$

to define the components of these tensors. In defining tensors, if there are obvious $\mathbb{Z}_2$ symmetries, we will often only give the non-zero components modulo these symmetries. Let GL be the *general linear group*; this is the group of all invertible linear transformations of V. If $\theta \in \otimes^k V^*$ and if $T \in \mathrm{GL}$, we define $T^*\theta \in \otimes^k V^*$ by:

$$(T^*\theta)(v_1, \ldots, v_k) := \theta(Tv_1, \ldots, Tv_k). \tag{1.1.a}$$

Similarly if $\theta \in \otimes^k V^* \otimes V$, we define

$$(T^*\theta)(v_1, \ldots, v_k) := T^{-1}\theta(Tv_1, \ldots, Tv_k). \tag{1.1.b}$$

There is a direct sum decomposition of $V^* \otimes V^*$ into irreducible modules where the structure group is the general linear group:

$$V^* \otimes V^* = \Lambda^2 \oplus S^2$$

as the sum of the *alternating tensors* Λ^2 of rank two and the *symmetric tensors* S^2 of rank two. If $\theta \in V^* \otimes V^*$, this decomposition yields $\theta = \theta_a + \theta_s$ where $\theta_a \in \Lambda^2$ and $\theta_s \in S^2$ are defined by setting:

$$\begin{aligned} \theta_a(x,y) &= \tfrac{1}{2}\{\theta(x,y) - \theta(y,x)\}, \\ \theta_s(x,y) &= \tfrac{1}{2}\{\theta(x,y) + \theta(y,x)\}. \end{aligned} \tag{1.1.c}$$

More generally, let Λ^k and S^k be the space of all *alternating* and *symmetric* tensors of degree k, respectively.

Fix a non-degenerate inner product $\langle \cdot, \cdot \rangle$ of *signature* (p, q) on V. We are in the *Riemannian setting* if $p = 0$ or, equivalently, if $\langle \cdot, \cdot \rangle$ is positive definite. Similarly, we are in the *Lorentzian* setting if $p = 1$. The *neutral setting* $p = q$ also is important. The pair $(V, \langle \cdot, \cdot \rangle)$ is called an *inner product space*. The associated *orthogonal group* $\mathcal{O} = \mathcal{O}(V, \langle \cdot, \cdot \rangle)$ is given by:

$$\mathcal{O} := \{T \in \mathrm{GL} : T^*\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle\}.$$

There is a natural extension of $\langle \cdot, \cdot \rangle$ to $\otimes^k V$ which will play a central role in our development and which we introduce here:

Definition 1.1.1 Let V^k denote the Cartesian product $V \times \cdots \times V$. If $\vec{v} = (v_1, \ldots, v_k)$ and $\vec{w} = (w_1, \ldots, w_k)$ are elements of V^k, the map

$$\vec{v} \times \vec{w} \to \langle v_1, w_1 \rangle \ldots \langle v_k, w_k \rangle$$

is a bilinear symmetric map from $V^k \times V^k$ to $\mathbb{R}$ which extends to a symmetric inner product that is the extension of $\langle \cdot, \cdot \rangle$ to $\otimes^k V$. If $\{e_i\}$ is an orthonormal basis for V and if $I = (i_1, \ldots, i_k)$ is a multi-index, let $e_I := e_{i_1} \otimes \cdots \otimes e_{i_k}$. The collection $\{e_I\}_{|I|=k}$ forms a basis for $\otimes^k V$ with

$$\langle e_I, e_K \rangle = \left\{ \begin{array}{rl} 0 & \text{if } I \neq K \\ \langle e_{i_1}, e_{i_1} \rangle \ldots \langle e_{i_k}, e_{i_k} \rangle & \text{if } I = K \end{array} \right\}.$$

Since $\langle e_I, e_I \rangle = \pm 1$, $\langle \cdot, \cdot \rangle$ is non-degenerate on $\otimes^k V$. The orthogonal group $\mathcal{O}$ extends to act naturally on $\otimes^k V$ and preserves this inner product.

We may use $\langle \cdot, \cdot \rangle$ to identify V with V^* and extend $\langle \cdot, \cdot \rangle$ to tensors of all types; the natural action of $\mathcal{O}$ on such tensors then preserves this inner product. For example, let $\varepsilon_{ij} := \langle e_i, e_j \rangle$ give the components of the inner product relative to an arbitrary basis $\{e_i\}$ (which need not be orthonormal) for V. The inverse matrix ε^{ij} then gives the components of the dual inner product on V^* relative to the dual basis $\{e^i\}$ for V^*:

$$\varepsilon^{ij} = \langle e^i, e^j \rangle.$$

The following is a useful identity that will play a central role in many of our calculations:

$$\varepsilon^{ij}\langle v, e_i\rangle e_j = v \quad \text{and} \quad \varepsilon^{ij}\langle e_i, e_j\rangle = m. \tag{1.1.d}$$

If $A \in \otimes^4 V^*$, we define Ricci contractions:

$$\begin{array}{ll} \rho_{12}(A)_{kl} := \varepsilon^{ij} A_{ijkl}, & \rho_{13}(A)_{jl} := \varepsilon^{ik} A_{ijkl}, \\ \rho_{14}(A)_{jk} := \varepsilon^{il} A_{ijkl}, & \rho_{23}(A)_{il} := \varepsilon^{jk} A_{ijkl}, \\ \rho_{24}(A)_{ik} := \varepsilon^{jl} A_{ijkl}, & \rho_{34}(A)_{ij} := \varepsilon^{kl} A_{ijkl}. \end{array} \tag{1.1.e}$$

We set $\rho = \rho_{14}$. These contractions are $\mathcal{O}$ but not GL invariants. Similarly, if $\mathcal{A} \in \otimes^2 V^* \otimes \operatorname{End}(V)$, we define:

$$\rho(\mathcal{A})_{jk} := \mathcal{A}_{ijk}{}^{i} = \operatorname{Tr}(z \to \mathcal{A}(z, x)y)\,;$$

this contraction does not depend on the inner product. We use Equation (1.1.c) to decompose $\rho = \rho_a + \rho_s$ as the sum of the *alternating Ricci tensor* and the *symmetric Ricci tensor*. The terminology that we will use is motivated by the geometric setting. Therefore, the trace of the Ricci tensor is called the *scalar curvature*; it is given by setting:

$$\tau := \varepsilon^{il}\varepsilon^{jk} A_{ijkl} = \varepsilon^{jk} \mathcal{A}_{ijk}{}^{i}.$$

Definition 1.1.2 Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space.

(1) We say that $J_- \in \mathrm{GL}$ is a *complex structure* on V if $J_-^2 = -\operatorname{Id}$; if in addition $J_-^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle$, then J_- is said to be a *pseudo-Hermitian complex structure* and the triple $(V, \langle\cdot,\cdot\rangle, J_-)$ is said to be a *pseudo-Hermitian vector space.* Such structures exist if and only if $(V, \langle\cdot,\cdot\rangle)$ has signature (p, q) where both p and q are even. The associated *Kähler form* is given by setting $\Omega_-(x, y) := \langle x, J_- y\rangle$. We shall often let $\Omega = \Omega_-$ when the context is clear.

(2) We say that $J_+ \in \mathrm{GL}$ is a *para-complex structure* if $J_+^2 = \operatorname{Id}$ and if $\operatorname{Tr}(J_+) = 0$. This latter condition is automatic in the complex setting, but must be imposed in the para-complex setting. If $J_+^*\langle\cdot,\cdot\rangle = -\langle\cdot,\cdot\rangle$, then J_+ is said to be a *para-Hermitian complex structure* and the triple $(V, \langle\cdot,\cdot\rangle, J_+)$ is said to be a *para-Hermitian vector space.* Such structures exist only in the *neutral signature* $p = q$. The associated *para-Kähler form* is given by setting $\Omega_+(x, y) := \langle x, J_+ y\rangle$. Again, we shall often set $\Omega = \Omega_+$.

If g is a smooth symmetric non-degenerate bilinear form on the tangent bundle TM of a smooth manifold M, then (M, g) is called a *pseudo-Riemannian manifold.* If J_- is an endomorphism of TM with $J_-^2 = -\operatorname{Id}$, then J_- is said to be an *almost complex structure* on M and the pair (M, J_-) is said to be an *almost complex manifold*; necessarily $m = 2\bar{m}$ is even. The classic integrability result (see [Newlander and Nirenberg (1957)]) is summarized in Theorem 3.4.2. We say that J_- is an *integrable complex structure* and that (M, J_-) is a *complex manifold* if the *Nijenhuis tensor*

$$N_-(x,y) := [x,y] + J_-[J_-x,y] + J_-[x,J_-y] - [J_-x,J_-y] \qquad \text{(1.1.f)}$$

vanishes or, equivalently, if in a neighborhood of any point of the manifold there are local *holomorphic coordinates* $(x^1, \dots, x^{\bar{m}}, y^1, \dots, y^{\bar{m}})$ so that we have $J_-\partial_{x_i} = \partial_{y_i}$ and $J_-\partial_{y_i} = -\partial_{x_i}$. If $J_-^*g = g$, then (M, g, J_-) is called an *almost pseudo-Hermitian manifold*; (M, g, J_-) is said to be a *pseudo-Hermitian manifold* if J_- is an integrable complex structure.

Similarly, following [Cortés et al. (2004)], we say that (M, J_+) is an almost para-complex manifold if J_+ is an endomorphism of TM such that $J_+^2 = \operatorname{Id}$ and $\operatorname{Tr}(J_+) = 0$; necessarily $m = 2\bar{m}$ is even. One says J_+ is an *integrable para-complex structure* if the *para-Nijenhuis tensor*

$$N_+(x,y) := [x,y] - J_+[J_+x,y] - J_+[x,J_+y] + [J_+x,J_+y] \qquad \text{(1.1.g)}$$

vanishes or, equivalently (see Theorem 3.4.3), if in a neighborhood of any point of the manifold there are local *para-holomorphic coordinates* $(x^1, \dots, x^{\bar{m}}, y^1, \dots, y^{\bar{m}})$ so that we have $J_+\partial_{x_i} = \partial_{y_i}$ and $J_+\partial_{y_i} = \partial_{x_i}$. If $J_+^*g = -g$, then (M, g, J_+) is said to be an *almost para-Hermitian manifold*; if J_+ is an integrable para-complex structure, then (M, g, J_+) is said to be a *para-Hermitian manifold.*

The vanishing of $N_\pm$ imposes additional curvature restrictions called the *Gray identity* that will be discussed presently in Theorem 1.9.1 in the complex and in the para-complex settings.

We present a few general purpose references which may provide basic background information in some areas and appologize in advance if your favorite is missing: [Besse (1987)], [Bourbaki (2005)], [Chevalley (1946)], [Cruceanu, Fortuny, and Gadea (1996)], [Eisenhart (1927)], [Eisenhart (1967)], [Evans (1998)], [Ferus, Karcher, and Münzer (1981)], [Frobenius (1877)], [Fukami (1958)], [Fulton and Harris (1991)], [García-Río, Kupeli, and Vázquez-Lorenzo (2002)], [Gilkey (2001)], [Iwahori (1958)], [Kobayashi and Nomizu (1969)], [Newlander and Nirenberg (1957)], [Nomizu (1956)],