INTRODUCTION TO HARMONIC ANALYSIS ON REDUCTIVE P-ADIC GROUPS

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MATHEMATICAL NOTES

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INTRODUCTION TO HARMONIC ANALYSIS ON REDUCTIVE P-ADIC GROUPS

Based on lectures by Harish-Chandra at
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by Allan J. Silberger

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FOREWORD

These notes represent the writer's attempt to organize and comprehend the mathematics communicated to him by Harish-Chandra, both in public lectures and private conversations, during the years 1971-1973. They offer the reader an ab initio introduction to the theory of harmonic analysis on reductive p-adic groups. Besides laying the foundations for a theory of induced representations by presenting Jacquet's theory, the Bruhat theory, the theory of the constant term, and the Maass-Selberg relations, these notes develop the theory of the Schwartz space on a p-adic group and the theory of the Eisenstein integral in complete detail. They also give the construction of the algebras of wave packets as orthogonal components of Schwartz space and prove Plancherel's formula for induced series, Harish-Chandra's commuting algebra theorem, and the sufficiency of the tempered spectrum for rank one groups. Most notable among omissions from these notes is Harish-Chandra's completeness theorem (i. e., that for arbitrary rank, the tempered spectrum suffices) announced in [7f] and his theory of the characters of admissible representations.

The reader will find a summary of a part of the contents of this work given in Harish-Chandra's Williams College lectures ([7e]).

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Chapter 0. On the Structure of Reductive p-adic Groups.

The theory to be developed in the five later chapters of these notes depends upon the structure theory for reductive groups with points in a p-adic field. Fortunately, this theory has been worked out in detail (cf., [2a], [2c] for the essentially algebraic aspects and [4c] for the directly related topological part). In this chapter we very briefly review only those facts from the structure theory which we shall need later. The reader may consult the references both for proofs and more details.

If G is any group and H a subgroup, we write $Z_G(H)[N_G(H)]$ for the centralizer [normalizer] of H in G. Given $x, g \in G$, we write $x^g = gxg^{-1}$ and $H^g = gHg^{-1}$.

We write $[X]$ for the cardinality of a finite set X.

For any ring R we write $R^\times$ for its group of units. If n is a positive integer and R is a commutative ring, we write $GL_n(R)$ for the group of all $n \times n$ matrices $(c_{ij})_{1 \leq i,\, j \leq n}$ with entries $c_{ij} \in R$ and determinant in $R^\times$. We write $\det(x)$ or $\det(c_{ij})$ to denote the determinant of a matrix x or (c_{ij}).

We write $\mathbb{Z}$ for the ring of ordinary (rational) integers, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ for the fields, respectively, of rational, real, and complex numbers.

§0.1. Some Definitions and Facts.

Let Ω be a field and $\bar{\Omega}$ an algebraic closure of Ω. For any positive integer m the space $\bar{\Omega}^m$ of all m-vectors with components in $\bar{\Omega}$ carries the Zariski and Ω topologies: A subset S of $\bar{\Omega}^m$ is called Zariski closed [Ω-closed] if S is the zero set of some finite set of polynomials in $\bar{\Omega}[x_1, \ldots, x_m]$[$\Omega[x_1, \ldots, x_m]$]. The complement of a Zariski closed [Ω-closed] subset of $\bar{\Omega}^m$ is termed Zariski open [Ω-open]. Obviously, the Zariski topology on $\bar{\Omega}^m$ is finer than the Ω-topology. A subset S_1 of a Zariski closed [Ω-closed] set $S_2 \subset \bar{\Omega}^m$ is called Zariski dense [Ω-dense] in S_2 if every Zariski closed [Ω-closed] subset of $\bar{\Omega}^m$ which contains S_1 also contains S_2.

Under an obvious identification, we may, for any positive integer n, regard $GL_n(\bar{\Omega})$ as the Ω-open subset $\det(x_{ij}) \neq 0$ of $\bar{\Omega}^{n^2}$ or, more conveniently for what follows, the Ω-closed subset $y \det(x_{ij}) = 1$ of $\bar{\Omega}^{n^2+1}$. A linear algebraic group or l. a. g. is a subgroup $\underset{\sim}{G} \subset GL_n(\bar{\Omega})$ which is Zariski closed as a subset of $\bar{\Omega}^{n^2+1}$. If $\underset{\sim}{G}$ is an l. a. g., the group law $(x, y) \mapsto x^{-1}y$ for $\underset{\sim}{G}$ is given by a polynomial mapping $\bar{\Omega}^{n^2} \times \bar{\Omega}^{n^2} \to \bar{\Omega}^{n^2}$. An l. a. g. $\underset{\sim}{G}$ is said to be defined over Ω or called an Ω-group if $\underset{\sim}{G}$ is Ω-closed in $\bar{\Omega}^{n^2+1}$ and if the group law is given by polynomials with coefficients in Ω.

We will denote linear algebraic groups by Roman letters with wavy underlines. If $\underset{\sim}{G}$ is an l. a. g. and R is a subring of $\bar{\Omega}$, we write $\underset{\sim}{G}(R)$ for $\underset{\sim}{G} \cap GL_n(R)$. If $\underset{\sim}{G}$ is an Ω-group, we sometimes write $G = \underset{\sim}{G}(\Omega)$.

A morphism of l. a. g. 's is a group homomorphism which is at the same time a polynomial mapping. An Ω-morphism is a morphism of Ω-groups in which the polynomials defining the mapping have coefficients in Ω.

An l.a.g. is called *connected* if it contains no l.a.g. as a subgroup of finite index. An l.a.g. $\underset{\sim}{G}$ contains a maximal connected subgroup, denoted $\underset{\sim}{G}^0$. If $\underset{\sim}{G}$ is an Ω-group, then so is $\underset{\sim}{G}^0$.

Given an l.a.g. $\underset{\sim}{G}$, we write $\underset{\sim}{X}(\underset{\sim}{G})$ for the group of all *rational characters* of $\underset{\sim}{G}$, i.e., the group of all morphisms of $\underset{\sim}{G}$ to $\underset{\sim}{GL}_1 = GL_1(\overline{\Omega}) = \overline{\Omega}^{\times}$. If $\underset{\sim}{G}$ is connected, $\underset{\sim}{X}(\underset{\sim}{G})$ is a free abelian group. If $\underset{\sim}{G}$ is an Ω-group, we write $X(G)$ for the subgroup of $\underset{\sim}{X}(\underset{\sim}{G})$ which consists of all Ω-morphisms. Under these conditions, if $\chi \in X(G)$, we frequently abuse language to regard χ as a homomorphism $\chi : G \to \Omega^{\times}$.

The *radical* $\underset{\sim}{R}_{\underset{\sim}{G}}$ [*unipotent radical* $\underset{\sim}{N}_{\underset{\sim}{G}}$] of an l.a.g. $\underset{\sim}{G}$ is the maximal connected normal solvable [unipotent] subgroup of $\underset{\sim}{G}$. If $\underset{\sim}{N}_{\underset{\sim}{G}} = (1)$, we call $\underset{\sim}{G}$ a *reductive group*. If $\underset{\sim}{R}_{\underset{\sim}{G}} = (1)$, we call $\underset{\sim}{G}$ a *semisimple group*. The derived group $\mathcal{D}\underset{\sim}{G}$ of a reductive group is semisimple; in fact, the radical of a reductive group is central and $\mathcal{D}\underset{\sim}{G} \cap$ (center of $\underset{\sim}{G}$) is finite.

A commutative, connected, and reductive l.a.g. is called a *torus*. The *dimension* of a torus is the rank of $\underset{\sim}{X}(\underset{\sim}{T})$. A torus which is an Ω-group is called an Ω-*torus*. The radical of a reductive Ω-group is the maximal Ω-torus lying in its center. If $\underset{\sim}{T}$ is an Ω-torus and T is isomorphic to the diagonal subgroup $D_n(\Omega) \subset GL_n(\Omega)$ for some n, then we call T an Ω-*split torus*; this is the case if and only if $\underset{\sim}{X}(\underset{\sim}{T}) = X(T)$. If $\underset{\sim}{T}$ is an Ω-torus, then there is a finite separable extension Ω' of Ω with respect to which $\underset{\sim}{T}$ is an Ω'-split torus.

A reductive Ω-group is called *anisotropic* (*over* Ω) if it contains no Ω-split torus.

§0.2. *Cartan Subgroups and Split Tori in Reductive Groups*.

For the remainder of the chapter assume that $\mathbf{G}$ *is a connected and reductive* Ω*-group*.

A *Cartan subgroup* of $\mathbf{G}$ is a maximal torus of $\mathbf{G}$. All Cartan subgroups of $\mathbf{G}$ are conjugate, hence have the same dimension. There exist Cartan Ω-subgroups of $\mathbf{G}$. The group $G = \mathbf{G}(\Omega)$ operates by conjugation on the set of Cartan Ω-subgroups of $\mathbf{G}$. If $\operatorname{char} \Omega = 0$, the number of orbits is finite--otherwise, it may be infinite.

The group G operates transitively by conjugation on the set of maximal Ω-split tori of $\mathbf{G}$. Let $\mathbf{Z}$ denote the maximal Ω-split torus lying in the center of $\mathbf{G}$. The dimension of a maximal Ω-split torus of the group $\mathbf{G}/\mathbf{Z}$ is called the *reduced*, *semisimple*, or *split rank* of $\mathbf{G}$.

For any Ω-group $\mathbf{X}$ a maximal Ω-split torus lying in the radical of $\mathbf{X}$ is called a *split component* of $\mathbf{X}$.

If $\mathbf{T}$ is an Ω-torus of $\mathbf{G}$, then $Z_{\mathbf{G}}(\mathbf{T})$ is a connected and reductive Ω-subgroup of $\mathbf{G}$. Thus, $\mathbf{T}$ is a split component of $Z_{\mathbf{G}}(\mathbf{T})$ if and only if $\mathbf{T}$ is the maximal Ω-split torus lying in the center of $\mathbf{T}$.

If $\mathbf{\Gamma}$ is a Cartan Ω-subgroup of $\mathbf{G}$, then $Z_{\mathbf{G}}(\mathbf{\Gamma})/\mathbf{\Gamma}$ is obviously anisotropic. In particular, if $\mathbf{\Gamma}/\mathbf{Z}$ is anisotropic, then $Z_{\mathbf{G}}(\mathbf{\Gamma}) = \mathbf{\Gamma}$.

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§0.3. Parabolic Subgroups of Reductive Groups.

A Borel subgroup $\underset{\sim}{B}$ of $\underset{\sim}{G}$ is a maximal connected solvable subgroup of $\underset{\sim}{G}$; equivalently, $\underset{\sim}{G}/\underset{\sim}{B}$ is a projective variety and $\underset{\sim}{B}$ is minimal with this property. All Borel subgroups of $\underset{\sim}{G}$ are conjugate. A parabolic subgroup of $\underset{\sim}{G}$ is a subgroup which contains a Borel subgroup. Every parabolic subgroup of $\underset{\sim}{G}$ is a connected l.a.g.

Let $\underset{\sim}{P}$ be a parabolic Ω-subgroup (or p-subgroup) of $\underset{\sim}{G}$. The unipotent radical $\underset{\sim}{N} = \underset{\sim}{N}_{\underset{\sim}{P}}$ of $\underset{\sim}{P}$ is also an Ω-subgroup of $\underset{\sim}{P}$. A connected and reductive Ω-subgroup $\underset{\sim}{M}$ of $\underset{\sim}{P}$ is called a Levi subgroup or Levi factor of $\underset{\sim}{P}$ if $\underset{\sim}{P} = \underset{\sim}{M} \cdot \underset{\sim}{N}$, an Ω-semidirect product of Ω-groups--this means, in particular, that $\underset{\sim}{P}$ is Ω-isomorphic to $\underset{\sim}{M} \times \underset{\sim}{N}$ as Ω-varieties. A subgroup $\underset{\sim}{M}$ of $\underset{\sim}{G}$ is a Levi subgroup of $\underset{\sim}{P}$ if and only if $\underset{\sim}{M} = Z_{\underset{\sim}{G}}(\underset{\sim}{A})$ for some split component $\underset{\sim}{A}$ of $\underset{\sim}{P}$. The group N acts transitively and freely on the set of split components of $\underset{\sim}{P}$, consequently, also on the set of Levi subgroups of $\underset{\sim}{P}$. The choice of a split component $\underset{\sim}{A}$ or Levi subgroup $\underset{\sim}{M} = Z_{\underset{\sim}{G}}(\underset{\sim}{A})$ determines a Levi decomposition $\underset{\sim}{P} = \underset{\sim}{M}\underset{\sim}{N}$ for $\underset{\sim}{P}$.

Let $(\underset{\sim}{P}, \underset{\sim}{A})$ be a parabolic pair or p-pair of $\underset{\sim}{G}$, i.e., a pair consisting of a parabolic Ω-subgroup $\underset{\sim}{P}$ of $\underset{\sim}{G}$ and a split component $\underset{\sim}{A}$ of $\underset{\sim}{P}$. The codimension of $\underset{\sim}{Z}$, the split component of $\underset{\sim}{G}$, in $\underset{\sim}{A}$ is called the parabolic rank or p-rank of $\underset{\sim}{P}$ or $(\underset{\sim}{P}, \underset{\sim}{A})$.

An Ω-split torus $\underset{\sim}{A}$ of $\underset{\sim}{G}$ is called a special torus of $\underset{\sim}{G}$ if $\underset{\sim}{A}$ is a split component of some p-subgroup of $\underset{\sim}{G}$. We note that $\underset{\sim}{A}$ is a special torus if and only if $\underset{\sim}{A}$ is the split component of $\underset{\sim}{M}_{\underset{\sim}{A}} = Z_{\underset{\sim}{G}}(\underset{\sim}{A})$. A special subtorus of a maximal Ω-split torus $\underset{\sim}{A}_0$ of $\underset{\sim}{G}$ is called an $\underset{\sim}{A}_0$-standard torus of $\underset{\sim}{G}$.

Given two p-pairs (P, A) and (P', A') of G, we write $(P, A) \succ (P', A')$ if $P \supset P'$ and $A' \supset A$. A p-pair (P_0, A_0) of G is called a minimal p-pair of G if P_0 is a minimal parabolic Ω-subgroup of G; in this case, A_0 is necessarily a maximal Ω-split torus of G. The pair (G, Z) is a p-pair; however, by convention we refer to a maximal proper p-pair, i.e., a p-pair of p-rank one, as a maximal p-pair.

Fix a minimal p-pair (P_0, A_0) of G. A p-pair (P, A) is called standard (with respect to (P_0, A_0)) if $(P, A) \succ (P_0, A_0)$, semistandard (with respect to A_0) if $A \subset A_0$. The set of semistandard p-pairs is finite and will be described in detail in §0.5. For any p-pair (P, A) there is one and only one standard p-pair (P_1, A_1) such that P is conjugate to P_1; in fact, there exists $x \in G$ such that $(P_1^x, A_1^x) = (P, A)$. In particular, any two minimal p-pairs are conjugate in this strong sense.

Let A be a special torus of G. Write $\mathcal{P}(A)$ for the set of parabolic Ω-subgroups of G with A as split component. The finitely many elements of $\mathcal{P}(A)$ are called associated (or A-associated) p-subgroups. §0.5 will also characterize $\mathcal{P}(A)$, when A is standard.

Let P_1 and P_2 be parabolic Ω-subgroups of G. We say that P_1 and P_2 are opposite parabolic subgroups of G if $P_1 \cap P_2$ is a Levi factor of both. For any $P \in \mathcal{P}(A)$ there is exactly one opposite parabolic subgroup $\overline{P} \in \mathcal{P}(A)$. In this case, we also say that (P, A) and $(\overline{P}, A)$ are opposite p-pairs.

Let (P, A) $(P = MN)$ be a standard p-pair of G. There is a one-one correspondence between [semi-standard] (standard) p-pairs (P', A') $(P' = M'N')$ of G such that $(P, A) \succ (P', A')$ and [semi-standard] (standard) p-pairs of M. This correspondence is defined as follows. Given (P', A') as above, set $({}^*\underset{\sim}{P}, \underset{\sim}{A}') = (\underset{\sim}{P}' \cap \underset{\sim}{M}, \underset{\sim}{A}')$. Then $({}^*\underset{\sim}{P}, \underset{\sim}{A}')$ is a p-pair of $\underset{\sim}{M}$ with the Levi decomposition ${}^*\underset{\sim}{P} = {}^*\underset{\sim}{M}\,{}^*\underset{\sim}{N} = \underset{\sim}{M}' \cdot \underset{\sim}{M} \cap \underset{\sim}{N}'$. For this correspondence, $({}^*\underset{\sim}{P}_0, \underset{\sim}{A}_0)$ $({}^*\underset{\sim}{P}_0 = \underset{\sim}{M}_0 \cdot {}^*\underset{\sim}{N}_0 = \underset{\sim}{M}_0 \cdot \underset{\sim}{N}_0 \cap \underset{\sim}{M})$ is the "standard" minimal p-pair of $\underset{\sim}{M}$.

§0.4. On the Rational Points of p-adic Reductive Groups.

A p-adic field is a topological field which, as a topological space, is a nondiscrete totally disconnected space in the sense of §1.1. Concretely, any p-adic field is a completion with respect to a discrete valuation of either a function field in one variable over a finite field of constants or of a number field. From here on, let Ω denote a p-adic field. We normalize the absolute value function on Ω such that, if O is any nonempty compact open subset of Ω, μ is a Haar measure on Ω, and $a \in \Omega^\times$, then $\mu(aO) = |a|\mu(O)$. In this case, if a is a prime element of Ω (i.e., if $|a| < 1$ and $|a|$ generates the group of values), then $|a| = q^{-1}$, where q is the module of Ω.

The space of m-vectors Ω^m has the product p-adic topology. It is easy to see that the intersection with Ω^m of any Ω-closed subset of $\overline{\Omega}^m$ is closed in the p-adic topology. In particular, the group of Ω-points of any Ω-group is a t.d. group (§1.2).

If $\underset{\sim}{X}$ is a ________ Ω-subgroup of $\underset{\sim}{G}$, then we call X a ________ subgroup of G. The blank can contain any of the terms defined in the preceding two sections. If $\underset{\sim}{T}$ is an Ω-split torus of $\underset{\sim}{G}$, we call T a split torus of G. If $(\underset{\sim}{P}, \underset{\sim}{A})$ is a p-pair of $\underset{\sim}{G}$ with $\underset{\sim}{P} = \underset{\sim}{M}\underset{\sim}{N}$ its associated Levi decomposition we call

(P, A) a *p-pair of* G and $P = MN$ its *Levi decomposition.*

All the groups $\underset{\sim}{X}$ considered in the previous two sections have the property that X is Zariski dense in $\underset{\sim}{X}$.

If $\underset{\sim}{G}$ is anisotropic, then G is compact.

Given $\chi \in X(G)$ and $x \in G$, we set $\langle \chi, H_G(x) \rangle = \log_q |\chi(x)|$. This defines a continuous homomorphism $H_G : G \to \mathrm{Hom}(X(G), \mathbb{Z})$. Set $^\circ G = \bigcap_{\chi \in X(G)} \ker|\chi| = \ker H_G$. Then $^\circ G$ is an open normal subgroup of G which contains every compact subgroup of G. Indeed, the factor group $G/{}^\circ G$ is a free abelian group.

Lemma 0.4.1. Let $\underset{\sim}{G}$ be a connected and reductive Ω-group and let $\underset{\sim}{Z}$ be the maximal Ω-split torus in the center of $\underset{\sim}{G}$. There is a natural injection $r^* : X(G) \hookrightarrow X(Z)$. The factor group $X(Z)/X(G)$ is finite.

Proof. The natural map r^* is restriction. We have only to show that r^* maps $X(G)$ injectively to a subgroup of the same rank as $X(Z)$. Note first that the semisimple subgroup $\mathcal{D}\underset{\sim}{G}$ is a connected normal Ω-subgroup of $\underset{\sim}{G}$. Set $\underset{\sim}{Z}' = \underset{\sim}{Z}/\underset{\sim}{Z} \cap \mathcal{D}\underset{\sim}{G}$ and $\underset{\sim}{G}' = \underset{\sim}{G}/\mathcal{D}\underset{\sim}{G}$. Then $\dim \underset{\sim}{Z}' = \dim \underset{\sim}{Z}$ and $\underset{\sim}{Z}'$ is a maximal Ω-split torus in $\underset{\sim}{G}'$. Since $X(G) = X(G') = X(Z')$, which is a subgroup of finite index in $X(Z)$, the lemma is true.

Corollary 0.4.2. The subgroup $^\circ G \cdot Z$ is of finite index in G.

Proof. Observe that $G/{}^\circ G \supset {}^\circ GZ/{}^\circ G$ and both groups are isomorphic to lattices of the same rank.

Remark. Let $G_1 = G \cap \mathcal{D}\underset{\sim}{G}$. If $\operatorname{char} \Omega = 0$, then $[G : G_1 \cdot Z] < \infty$; however, if char $\Omega > 0$, this is not always true. Note that ${}^{o}G$ is not necessarily the group of Ω-rational points of an l. a. g. (e. g., ${}^{o}\Omega^{\times} = \{x \in \Omega^{\times} \mid |x| = 1\}$). However, $G/G_1 Z$ is compact and abelian.

§0. 5. Lie Algebras, Roots, and Weyl Groups.

Let A be a special torus of G and let $M = Z_G(A)$. Then M is a Levi subgroup of P for all $P \in \mathcal{P}(A)$. We define the Weyl group of A (relative to G) as $W(G/A) = W(A) = N_{\underset{\sim}{G}}(\underset{\sim}{A})/Z_{\underset{\sim}{G}}(\underset{\sim}{A}) = N_G(A)/Z_G(A)$. We note that $Z_{\underset{\sim}{G}}(\underset{\sim}{A}) = N_{\underset{\sim}{G}}(\underset{\sim}{A})^{o} = \underset{\sim}{M}$, which implies that $W(A)$ is a finite group. More generally, if A_1 and A_2 are special tori, we write $W(A_2 | A_1)$ for the set of homomorphisms $s : A_1 \to A_2$ which are induced by inner automorphisms of G.

There is a natural action of $W(A)$ on A and, dually, on $X(A)$. Given $a \in \underset{\sim}{A}$ and $s \in W(A)$, set $s \cdot a = a^{s} = a^{y} = yay^{-1}$, where $y = y(s) \in N_{\underset{\sim}{G}}(\underset{\sim}{A})$ represents s; for $\chi \in X(A)$ define χ^{s} such that $\chi^{s}(s \cdot a) = \chi(a)$ $(a \in \underset{\sim}{A})$.

Define the real Lie algebra of A as $\mathfrak{a} = \mathrm{Hom}(X(A), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ and its dual $\mathfrak{a}^{*} = X(A) \otimes_{\mathbb{Z}} \mathbb{R}$, also the complexifications $\mathfrak{a}_{\mathbb{C}} = \mathfrak{a} \otimes \mathbb{C}$ and $\mathfrak{a}^{*}_{\mathbb{C}} = \mathfrak{a}^{*} \otimes \mathbb{C}$. Notice that, canonically, $\mathfrak{a} = \mathrm{Hom}(X(M), \mathbb{Z}) \otimes \mathbb{R}$ and $\mathfrak{a}^{*} = X(M) \otimes \mathbb{R}$. The mapping H_M of §0. 4 imbeds $M/{}^{o}M$ as a lattice in $\mathfrak{a}$; each element $\chi \in X(A)$ corresponds to a unique element of $\mathfrak{a}^{*}$, called the associated weight. We usually denote rational characters and the canonically associated weights by the same Greek letter, depending upon the context to indicate the intended meaning. The pairing $\langle \, , \, \rangle$ of §0. 4 extends to a pairing of $\mathfrak{a}^{*} \times \mathfrak{a}$ to $\mathbb{R}$, of $\mathfrak{a}^{*}_{\mathbb{C}} \times \mathfrak{a}_{\mathbb{C}}$ to $\mathbb{C}$. The group $W(A)$ operates on both $\mathfrak{a}$ and $\mathfrak{a}^{*}$.

Next we let the action of $\underset{\sim}{G}$ on $\underset{\sim}{G}$ by inner automorphisms induce the adjoint representation $\mathrm{Ad} : \underset{\sim}{G} \to \mathrm{Aut}(\underset{\sim}{\mathfrak{g}})$, where $\underset{\sim}{\mathfrak{g}}$ is the Lie algebra of $\underset{\sim}{G}$ and $\mathrm{Aut}(\underset{\sim}{\mathfrak{g}})$ is its automorphism group. The group $\mathrm{Aut}(\underset{\sim}{\mathfrak{g}})$ is an Ω-group and Ad is an Ω-morphism. We write $\mathrm{Ad}_{\underset{\sim}{A}}$ for the restriction of Ad to $\underset{\sim}{A}$. Since $\underset{\sim}{A}$ is Ω-split, $\mathrm{Ad}_{\underset{\sim}{A}}$ is diagonalizable (over Ω). We call a nontrivial rational character of $\underset{\sim}{A}$ which occurs in $\mathrm{Ad}_{\underset{\sim}{A}}$ a <u>root character</u>. The weights of the root characters are called the <u>roots</u> or A-<u>roots</u> with respect to G. An A-root is called <u>reduced</u> if $t\alpha$ an A-root with $t \in \mathbb{Q}$ implies $t \in \mathbb{Z}$. We write $\Sigma(\underset{\sim}{G}, \underset{\sim}{A})$ or $\Sigma(G, A)$ $[\Sigma_r(\underset{\sim}{G}, \underset{\sim}{A})$ or $\Sigma_r(G, A)]$ for the set of A-roots [reduced A-roots] with respect to G. We have the direct sum decomposition $\underset{\sim}{\mathfrak{g}} = \underset{\sim}{\mathfrak{m}} \oplus \underset{\sim}{\mathfrak{g}}_\alpha$ $(\alpha \in \Sigma(G, A))$, where $\underset{\sim}{\mathfrak{m}}$ is the Lie algebra of $\underset{\sim}{M}$ and $\underset{\sim}{\mathfrak{g}}_\alpha$ the eigenspace in $\underset{\sim}{\mathfrak{g}}$ associated to the root character $\alpha \in X(A)$.

To each pair $\pm\alpha \in \Sigma_r(G, A)$ there corresponds an orthogonal hyperplane $H_\alpha = \{a \in \mathfrak{a} \mid \langle\alpha, a\rangle = 0\}$. The connected components of the space $\mathfrak{a} - \cup H_\alpha = \mathfrak{a}'$ $(\alpha \in \Sigma(G, A))$ are called <u>chambers</u>. Choosing a chamber $\mathfrak{C} \subset \mathfrak{a}$, we obtain a set $\Sigma_{\mathfrak{C}} = \{\alpha \in \Sigma(G, A) \mid \langle\alpha, \mathfrak{C}\rangle > 0\}$. There is also a unique set $\Sigma^o_{\mathfrak{C}} \subset \Sigma_{\mathfrak{C}}$ of <u>simple roots</u> such that the elements of $\Sigma^o_{\mathfrak{C}}$ are linearly independent and every element of $\Sigma_{\mathfrak{C}}$ is a positive integer combination of elements of $\Sigma^o_{\mathfrak{C}}$. Let $\underset{\sim}{\mathfrak{n}} = \oplus\, \underset{\sim}{\mathfrak{g}}_\alpha$ $(\alpha \in \Sigma_{\mathfrak{C}})$. Then $\underset{\sim}{\mathfrak{n}}$ is the Lie algebra of the unipotent radical $\underset{\sim}{N}$ of a parabolic subgroup $P = MN$ of G. We also write $\Sigma(P, A) = \Sigma_{\mathfrak{C}}$, $\Sigma^o(P, A) = \Sigma^o_{\mathfrak{C}}$ and $\Sigma_r(P, A) = \Sigma_r(G, A) \cap \Sigma_{\mathfrak{C}}$. Observe that $-\Sigma_{\mathfrak{C}} = \Sigma_{-\mathfrak{C}}$ and that $\Sigma(G, A) = \Sigma_{\mathfrak{C}} \cup \Sigma_{-\mathfrak{C}}$. The chamber $-\mathfrak{C}$ corresponds to the opposite parabolic subgroup $\overline{P} = M\overline{N} \in \mathcal{P}(A)$. We have the following one-one correspondences:

$$\mathcal{P}(A) \leftrightarrow \{\mathfrak{C} \mid \mathfrak{C} \subset \mathfrak{a}\} \leftrightarrow \{\Sigma_{\mathfrak{C}} \mid \Sigma_{\mathfrak{C}} \subset \Sigma(G, A)\} \leftrightarrow \{\Sigma^o_{\mathfrak{C}}\}.$$

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