Energy Methods in Electromagnetism

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Preface

Electrical devices and systems are generally described either in terms of circuit parameters or in terms of fields distributed in space. This book deals with the relationship between fields and circuits, a subject which seems to have received relatively little attention and which has puzzled me greatly for many years. When I began to think about it, I found that energy could be used as a unifying principle, but I had to study variational mechanics before I saw that the concept I was looking for was 'energy at equilibrium'. Although my exploration of this subject is far from complete I hope that this book will be found helpful by the reader who is himself exploring these matters.

To some people this talk of exploration will sound strange. I have heard it said that electromagnetism is completely closed as a subject, because Maxwell's equations define all that can be known about it. Although I have the utmost respect for these equations and for the man who discovered them, I am sure that electromagnetism is far from closed. Maxwell's equations are an invitation to travel abroad, just as maps are, and there is a good deal of difference between looking at a map and travelling in foreign countries. Even a cursory reading of Maxwell's *Electricity and Magnetism* will show that this work is more like an explorer's diary than an encyclopaedia.

Some readers will feel that, while this may be so, the proof of the pudding is in the eating and that the improved understanding which they may gain from this book must be matched by improved methods of calculation and design. I agree with that sentiment and hope that the book will be useful in this respect also. There are a considerable number of worked examples, which should be sufficient to enable the interested reader to apply the energy method to problems within his own speciality.

My thanks are due to many people, including all those who, when they came to see me on some other matter, found themselves subjected to a long discourse on electromagnetism and bore it cheerfully. By name I can mention only a few: the late Professor E. B. Moullin, who first shared his enthusiasm for electromagnetic problems with me some thirty years ago, Dr. G. K. Cambrell who introduced me to the algebra of vector spaces, Dr. F. J.

Evans, a fellow-explorer who first told me of Lanczos' *Variational Principles of Mechanics*, Professor M. J. Sewell, who sent me his valuable papers on convex and concave functionals, Dr. J. Penman who helped in the preparation of two papers published in the *Proceedings* of the I.E.E., Professor G. Rodriguez-Izquierdo who criticised incisively the methods proposed in those papers, Drs. J. B. Davies, D. R. Farrier, and P. J. Tavner who read the typescript and made many valuable comments, and Dr. R. L. Stoll, friend and fellow-worker for many years. I also want to thank Miss S. D. Makin for her patient and expert typing and my wife for suffering electromagnetism to invade our house, while this book was being written.

P.H.

Southampton
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List of principal symbols

A	action
$\mathbf{A}$	magnetic vector potential
$\mathbf{B}$	magnetic flux density
C	capacitance
$\mathbf{C}$	vector function
$\mathbf{D}$	electric flux density
$\mathbf{E}$	electric field strength
E	electromotive force
$\mathbf{F}$	force
G	content, Green's function
$\mathbf{H}$	magnetic field strength
I	electric current
$\mathbf{J}$	electric current density
J	moment of inertia
L	Lagrangian energy, inductance, operator
M	mutual inductance
P	power
$\mathbf{P}$	vector function
$\mathbf{Q}$	vector function
Q	electric charge
Q^*	magnetic pole strength
R	resistance, reactive power
T	kinetic energy, torque
U	potential energy
V	electric potential, potential difference
W	energy
X	reactance
Y	convex energy functional
Z	concave energy functional

$\mathbf{a}$	acceleration
b	damping constant
f	force, basis function
g	conductance, source distribution
k	spring constant
m	mass
p	direct stress, general momentum
q	general coordinate
$\dot{q}$	general velocity
r	resistance
t	time
$\mathbf{v}$	velocity
w	work function
x	reactance
γ	propagation constant
ε_0	permittivity of free space
ε_r	relative permittivity
ε	$\varepsilon_0\varepsilon_r$
$\lambda, \boldsymbol{\lambda}$	Lagrange's multiplier
μ_0	permeability of free space
μ_r	relative permeability
μ	$\mu_0\mu_r$
ρ	volume charge density
ρ_m	volume pole density
σ	surface charge density
τ	shear stress
ϕ	electric scalar potential
ϕ_m	magnetic scalar potential
χ	scalar potential
ψ	scalar potential
ω	angular frequency
Δ	eddy current skin depth
Φ	magnetic flux
Ψ	electric flux

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1. Electromagnetism and Mechanics

1.1. Introduction

The purpose of this book is to show that a method of looking at electromagnetic phenomena from the standpoint of energy has much to recommend it. The three advantages which arise from this approach are first that the conceptual framework of the subject is simplified, second that particularly simple methods of calculation can be devised from energy considerations, and third that these methods of calculation are directly related to measurement techniques. In the book we shall concentrate chiefly on the first and second advantages, namely on concepts and calculations. Since the subject of electromagnetism is vast in extent and since the author's experience is limited, the book is intended as an introduction to a point of view rather than as a handbook of useful results. Nevertheless it is hoped that the reader will find sufficient help to enable him to apply the method to his own field of interest.

We must of course admit straight away that there is nothing radically new in looking at electromagnetism from an energy point of view. All books on the subject include energy theorems such as Thomson's theorem, which we shall discuss in detail in Chapter 4. However, what is perhaps different about this book is that we regard these theorems as central, whereas energy methods are generally treated as being somewhat peripheral to the main development. An example will illustrate what we have in mind. Suppose we think of electrostatic problems, which are often considered to require the calculation of the field at every point of the region of interest. In general this requires a subdivision of such a region into a fine mesh of field parameters which have to be calculated by means of a numerical scheme. The calculation is complete when all the local values have converged to the actual values. Central to such a scheme is the concept of a potential function at a point. It is our contention that this may be a wasteful method because it generates information which is unlikely to be of use either to the manufacturer or the operator of a device. It is very seldom that anybody requires to know a potential distribution. Such a distribution would have to be measured with a very fine probe and even then a point function is unobtainable. Generally what is required is a knowledge of the energy distribution, expressed perhaps in terms of capacitance, and a fairly coarse distribution will be sufficient. The energy method of this book deliberately restricts the information to that which is required. Instead of focusing attention on a point potential it works with the energy of the whole device treated as a system. By noting that the system energy at equilibrium has a maximum or minimum value we can obtain relatively simple approximations to the energy distribution. These approximations provide both upper and lower bounds to the accurate values and so establish confidence limits which can be set by the engineer in accordance with his needs in any particular application.

What has been said here about an electrostatic problem has far wider applications. It applies equally well to magnetostatics and to problems concerned with electromagnetic induction and radiation. Equally well it applies to network problems, where the energy distribution is often more important than a knowledge of all the individual currents and voltages. From a measurement point of view it is of course apparent that the energy level is the critical factor.

We hope that these practical considerations of calculation and measurement will encourage the reader to study the method. It hardly needs saying that energy methods are unlikely to displace other methods of calculation. All that they will do is to increase the choice of method and to reduce the complexity of a calculation in appropriate cases. This means that the energy method can be applied to problems of complicated geometry and to problems containing non-linear parameters which present particular difficulties in field calculations. It also means that less complex problems can be transferred from a large digital computer to a desk machine with a corresponding reduction in time and cost. This might be especially useful in the early stages of a design process and could be followed by a more detailed and accurate computation when the design is nearing completion.

In this introduction it is impossible to develop a convincing argument that the energy approach to electromagnetism simplifies the conceptual structure of the subject. This will become apparent in subsequent chapters. To the author, who is primarily a teacher, the conceptual simplicity of the approach is of overriding importance. In popular usage theory is often contrasted with practice, but in science and engineering theory and practice are closely joined. The use of design formulae without theoretical backing is unsatisfactory and dangerous. At best it can be justified where the design of a device is well established, but when changes have to be made it is necessary to go back to first principles. It is then of the greatest importance that these principles should be as simple as possible, not only because this will enable an individual designer to think out improvements, but also in order to provide a simple language in which ideas can be interchanged.

For this reason much of this book is concerned to provide the reader with an understanding of the physical principles of electromagnetism which underlie the method of calculation. The principles are few in number and can be understood without much difficulty. A grasp of the principles will enable us to compare different processes and to obtain quickly a qualitative understanding of particular devices. Such understanding results in a great saving of mental effort and enables new developments to be incorporated into one's mental framework of ideas. This provides mental stability, which is a valuable possession especially in dealing with a rapidly developing subject.

The needs and interests of different readers will vary. It is a convention that a book should be read from beginning to end and in a detective story one should resist the temptation to turn to the last chapter to find the solution. However, in a book like this there is no need to proceed in any particular manner. Learning is generally a cyclical process. To understand a matter we need to look at it many times and from different angles. The reader who is interested chiefly in application may find it best to turn at once to Chapters 4–6 and later to Chapters 2 and 3, but the reader who is interested chiefly in the structure of the subject may find it preferable to follow the order of the book. The level of treatment is such that it should be suitable for students in the final year of a degree course or during postgraduate studies, but the book is intended also for practising engineers, especially for those concerned with the design and analysis of electromagnetic devices.

1.2. Historical background

We have already mentioned that this book does not need to be read in strict sequence, and this is particularly true with reference to this section which can easily be omitted on a first reading. The section will interest only those readers who, like the author, are fascinated by the history of ideas. How did the notions of electric charge and field arise and how is electromagnetism related to other branches of physical science or to science in general? Answers to these questions should give insight into the use of the various concepts. We have inherited a toolkit of ideas and may be helped in the selection of appropriate tools if we consult the people who developed them in the first instance. Although the ideas underlying the subject are few in number, so that the language of electromagnetism employs a relatively small number of words, this certainly does not mean that the construction of this language was a simple matter. Indeed the opposite is true and we have inherited something which is the result of the labours of men of genius over a period of many centuries. A historical approach, although it is somewhat out of fashion, will help us to gain a perspective and will also anchor new developments in past experience. It is of course true that the scaffolding used in construction should be removed to show the beauty of the completed building, but the analogy is misleading. In the development of physical concepts the scaffolding and the building are the same thing. Insight into the meaning of concepts can best be acquired by looking at the way in which these concepts were selected by the great pioneers of the subject.

Although the Babylonians and Egyptians made progress in technology and the practical application of mechanics, they did not formulate general principles. It was the genius of the Greeks to be particularly concerned with giving a coherent account of the phenomena around them and most of our present scientific ideas can be traced to Greek thought.

The first great name in Greek science is that of Thales of Miletus (around 600 B.C.). To him is attributed the discovery of electrostatic attraction. The word electricity is derived from the Greek word for amber, a substance which was used in these early experiments. It is interesting that a subject which was to be developed only in modern times should have its roots right at the beginning of history. More important than Thales' observation of electrostatic effects was his contribution to scientific thought. He postulated that the multitude of physical phenomena must be related to a single invariant entity. With this bold conjecture he charted the course which science has taken ever since, namely the search for invariants, or conservation laws, such as the conservation of matter or of energy. By this means scientists have been able to reduce the number of hypotheses and devise laws of ever-increasing generality. Thales chose water as his universal substance or principle. His contemporaries Anaximenes and Anaximander accepted the idea but were less specific. The former chose air, and the Greek word could also mean breath or spirit. The latter deliberately avoided the name for his substance which was the substratum of all that could be observed. He might well have been satisfied with our present idea of a space–time continuum.

Having defined an invariant substance these philosophers explained observed change in terms of the motion of substance. Motion was taken as a basic idea not needing further explanation. Ever since the time of these three philosophers of Miletus matter and motion have been the foundation stones of physical science.

The idea of force as being the cause of motion was developed a century latter by Empedocles. He defined both attractive and repulsive forces and described centrifugal force. These ideas fit well into the modern view of a universal gravitational attraction and cosmic repulsion. Attractive and repulsive forces are also dominant in electrostatics and magnetostatics. Another of Empedocles' important contributions was the 'exclusion principle' that two bodies cannot be in the same place.

Empedocles also taught that light is propagated through space with a finite velocity. He generalized the idea into a theory that all bodies give off some of their substance as emanations, and he used this as an explanation of the attractive force of a magnet on iron. This is an idea we find again in William Gilbert's book on magnetism published in 1600 A.D.

The Greek thinker who had the most lasting and dominant influence on scientific thought was Aristotle who lived in the fourth century B.C. His scientific views were accepted very widely until the time of the Renaissance and the start of modern science. At that time the settled Aristotelian world-view was a great hindrance to progress. Aristotle, like the other Greek philosophers, was strong in the realm of ideas but did not in general submit the ideas to the test of experiment. The Greeks made little progress in technology and so their science lacked the stimulus of accurate observation. However, they were pre-eminent in conceptual thinking and it would be foolish to dismiss Aristotle's contributions in this realm.

The chief difference between Aristotle's views and our own arises from Aristotle's preoccupation with biological rather than physical ideas. The modern view of physics, which has been so remarkably successful, is a mechanistic view. We confine ourselves to the question 'how' the universe works and deliberately refuse to introduce the questions of purpose prefixed by 'why'. The idea of personality is kept at arm's length and all is impersonal matter and motion. Even the study of living organisms is dominated by this approach. This does not commit us to the view that personality is a fiction, but it means that it is not a term of physical science. Nor is it a term of any other branch of science, if these branches are thought of as subsystems of physics, or at least of being subject to the methodology of physics.

Aristotle, on the other hand, makes purpose the dominant concept in his scheme of things. The regularity of nature which we ascribe to mechanical laws he sees as the purposeful action of mind. This preoccupation superposes an additional layer of concepts on physics and complicates science. Aristotle attempted to solve both physical and metaphysical questions with one scheme and in this optimistic attempt he was bound to fail. Nevertheless it is very worthwhile to take a look at his ideas.

Aristotle's theory of motion is of particular interest. He distinguishes between 'natural' and 'compulsory' motion. The former is either circular or linear. The heavenly bodies move in circular motion because this motion is simple and symmetrical. Circular motion does not need to vary and this makes it pre-eminent. There is also natural linear motion: light bodies move upward and heavy bodies downward. They do not need a force to move them, because force has been replaced by the concept of weight. Aristotle concluded wrongly that the velocity of a falling body is proportional to its weight.

All other motion is compulsory and needs force, the distance traversed by a body being proportional to the force and inversely proportional to the mass of the body. About 2000 years elapsed before these laws of motion were put on to a satisfactory basis by Galileo and Newton. This testifies to the difficulty of the problem. The ideas of natural and compulsory motion have been abandoned, but it is interesting to observe that they reflect the two aspects of mass, namely its gravitational and inertial properties. It is only in Einstein's general relativity theory that these two separate properties have been unified.

One other interesting idea of Aristotle occurs in his discussion of the impossibility of a vacuum. He does not allow the distinction between geometry and matter. If an object were to be surrounded by vacuum it would in accordance with his ideas have no interaction with anything else and it would be impossible to ascribe to it a position or place with respect to the rest of the world. This is a denial of the possibility of 'action at a distance' stated in the strongest possible form. It makes geometry into a feature of matter instead of regarding it as an abstract framework which determines the relative position of material objects. This theory represents a remarkable foreshadowing of the modern view in which the metric of space–time is dependent on the presence of matter. However, even before the formulation of general relativity theory Aristotle's view of action through contact had a profound influence. It was the guiding principle of the world picture of Descartes who postulated that space was filled with a substance which, although it could not be observed directly, was capable of transmitting force and energy. This substance was given the Greek name aether, a word meaning the upper air. Aether theories have been prominent in the development of the scientific description of electricity and magnetism, although the word itself has dropped out of use. The present-day ideas of electromagnetic fields are closely related to this concept.

Aristotle's views were challenged by the Greek atomist school, whose teaching is associated with the name of Epicurus, a younger contemporary of Aristotle. The atomist universe consisted of a vacuum filled with solid particles. These particles were of the same substance but had different shapes. All these atoms were thought to be in a state of permanent violent movement. Force was transmitted by impact and action at a distance was explicitly denied. Even vision was due to the impact on our eyes of material emanations from the objects which the eyes see. The relationship between Greek and modern atomic theory is fascinating but outside our present interest of tracing the development of electromagnetic and mechanical ideas.

After the brilliance of the Greek period there came centuries during which scientific thought was virtually at a standstill. There are almost two thousand years between Aristotle and the beginning of the modern period of rapid development. The great puzzle of motion was solved by the combined efforts of Galileo and Newton. Where Aristotle had postulated that force was proportional to velocity, a view which could be supported by observation of a horse and cart, Galileo postulated that force was proportional to change of velocity. It is easy to underestimate the revolutionary nature of this difference. It involved the idea of motion without force or contact, motion in an empty Euclidean space stretching to infinity. The idea of inertia, which now seems obvious, would have been to a follower of Aristotle a tangle of absurd notions.

Newton's gravitational theory (1687) produced another great change of outlook. The idea of a universal force of gravitational attraction depending only on the distance between material objects effectively separated geometry from matter. Newton's theory is the first example of a theory of action at a distance. The great advantage of such theories is that they enable attention to be focused on individual objects, whereas the continuum theories look at the whole universe as a single entity. Newton himself had misgivings about action at a distance as being the complete story. He even went so far as to call it an absurdity and suggested that his theory did not touch on the mechanism of gravitation but only gave a mathematical description of the phenomenon. However, in spite of this bow in Aristotle's direction Newton's theory marked a new point of departure. In particular his method was radically different from that of Descartes which was dominant in France. Voltaire wrote humorously in 1730: 'A Frenchman who arrives in London will find Philosophy, like everything else, very much changed there. He had left the world a plenum, and now finds it a vacuum.'

The history of electricity and magnetism followed a parallel development. William Gilbert's important book *De Magnete* published in 1600 ascribes electrostatic forces to emanations from electrified bodies. These emanations were material substances but of a very tenuous nature. Gilbert compared them to a scent which could be given out by a body for a long time without causing an appreciable loss of weight. He was less definite about magnetic forces because these were much stronger than electric ones. He ascribed them to the fact that magnets were surrounded by a region of magnetic strength, as we should say by a magnetic field. Gilbert explored the field with a small magnetic needle and noted the direction as well as the strength of the vector field of force. He also drew attention to the importance of the poles of his magnets and postulated the idea that the earth was itself a great dipole.

Gilbert's theory was a continuum theory. Like all such field theories it was difficult to quantify and lacked the simplicity of Newton's gravitational formula. It was not until electricity and magnetism were brought into the Newtonian framework in the middle of the 18th century, about 150 years after Gilbert, that progress became rapid. The way in which this came about is interesting.

Experiments with friction machines suggested that there were two classes