Graduate Texts in Mathematics

C.T.J.Dodson
T.Poston

Tensor Geometry

The Geometric Viewpoint and its Uses

Second Edition

张量几何

第2版

Springer

世界图书出版公司
www.wpcbj.com.cn

C. T. J. Dodson T. Poston

Tensor Geometry

The Geometric Viewpoint and its Uses

With 177 Figures

Second Edition

Springer

图书在版编目（CIP）数据

张量几何：第2版 = Tensor Geometry: The Geometric Viewpoint and its Uses 2nd ed：英文/（英）多德森（Dodson，C. T. J.）著．—北京：世界图书出版公司北京公司，2009.6

ISBN 978-7-5100-0479-7

Ⅰ.张… Ⅱ.多… Ⅲ.张量—研究生—教学参考资料—英文 Ⅳ.0183.2

中国版本图书馆 CIP 数据核字（2009）第084723号

书　　名： Tensor Geometry: The Geometric Viewpoint and its Uses 2nd ed.
作　　者： C. T. J. Dodson & T. Poston

中 译 名： 张量几何 第2版
责任编辑： 高蓉　刘慧

出 版 者： 世界图书出版公司北京公司
印 刷 者： 三河国英印务有限公司
发　　行： 世界图书出版公司北京公司（北京朝内大街137号　100010）
联系电话： 010-64021602，010-64015659
电子信箱： kjb@wpcbj.com.cn

开　　本： 24开
印　　张： 19
版　　次： 2009年06月
版权登记： 图字：01-2009-1104

书　　号： 978-7-5100-0479-7/O·694　　**定　　价：** 50.00元

Graduate Texts in Mathematics 130

Springer
Berlin
Heidelberg
New York
Barcelona
Budapest
Hong Kong
London
Milan
Paris
Santa Clara
Singapore
Tokyo

Preface to the Second Printing of the Second Edition

This edition is essentially a reprinting of the Second Edition, with the addition of two items to the Supplementary Bibliography, namely, Dodson and Parker: A User's Guide to Algebraic Topology, and Gray: Modern Differential Geometry of Curves and Surfaces.

This latter text is very important since it contains Mathematica programs to perform all of the essential differential geometric operations on curves and surfaces in 3-dimensional Euclidean space. The programs are available by anonymous ftp from bianchi.umd.edu/pub/ and are being used as support for a course at, among other places, UMIST: http://www.ma.umist.ac.uk/kd */ma351/ma351.html* .

June 1997

Kit Dodson
Manchester, U.K.

Tim Poston
Singapore

Preface to the Second Edition

We have been very encouraged by the reactions of students and teachers using our book over the past ten years and so this is a complete retype in TEX, with corrections of known errors and the addition of a supplementary bibliography. Thanks are due to the Springer staff in Heidelberg for their enthusiastic support and to the typist, Armin Köllner for the excellence of the final result. Once again, it has been achieved with the authors in yet two other countries.

November 1990

Kit Dodson
Toronto, Canada

Tim Poston
Pohang, Korea

Introduction

The title of this book is misleading.

Any possible title would mislead somebody. "Tensor Analysis" suggests to a mathematician an ungeometric, manipulative debauch of indices, with tensors ill-defined as "quantities that transform according to" unspeakable formulae. "Differential Geometry" would leave many a physicist unaware that the book is about matters with which he is very much concerned. We hope that "Tensor Geometry" will at least lure both groups to look more closely.

Most modern "differential geometry" texts use a coordinate-free notation almost throughout. This is excellent for a coherent understanding, but leaves the physics student quite unequipped for the physical literature, or for the specific physical computations in which coordinates are unavoidable. Even when the relation to classical notation is explained, as in the magnificent [Spivak], *pseudo*-Riemannian geometry is barely touched on. This is crippling to the physicist, for whom spacetime is the most important example, and perverse even for the geometer. Indefinite metrics arise as easily within pure mathematics (for instance in Lie group theory) as in applications, and the mathematician should know the differences between such geometries and the positive definite type. In this book therefore we treat both cases equally, and describe both relativity theory and (in Ch. IX, §6) an important "abstract" pseudo Riemannian space, SL(2;R).

The argument is largely carried in modern, intrinsic notation which lends itself to an intensely geometric (even pictorial) presentation, but a running translation into indexed notation explains and derives the manipulation rules so beloved of, and necessary to, the physical community. Our basic notations are summarised in Ch. 0, along with some basic physics.

Einstein's system of 1905 deduced everything from the Principle of Relativity: that no experiment whatever can define for an observer his "absolute speed". Minkowski published in 1907 a geometric synthesis of this work, replacing the once separately absolute space and time of physics by an absolute four dimensional spacetime. Einstein initially resisted this shift away from argument by comparison of observers, but was driven to a more "spacetime geometric" view in his effort to account for gravitation, which culminated in 1915 with General Relativity. For a brilliant account of the power of the Principle of Relativity used directly, see [Feynman]; particularly the deduction (vol. 2, p. 13–16) of magnetic effects from the laws of electrostatics. It is harder to maintain this approach when dealing with the General theory. The Equivalence Principle (the most physical assumption used) is hard even to state precisely without the geometric language of covariant differentiation, while Einstein's Equation involves sophisticated geometric objects. Before any detailed physics, therefore, we develop the geometrical setting: Chapters I – X are a geometry text, whose material is chosen with an eye to physical usefulness. The motivation is largely geometric also, for accessibility to mathematics students, but since physical thinking occasionally offers the most direct insight into the geometry, we cover in Ch. 0, §3 those elementary facts about special relativity that we refer to before Ch. XI. British students of either mathematics or physics should usually know this much before reaching university, but variations in educational systems – and students – are immense.

The book's prerequisites are some mathematical or physical sophistication, the elementary functions (log, exp, cos, cosh, etc.), plus the elements of vector algebra and differential calculus, taught in any style at all. Chapter I will be a recapitulation and compendium of known facts, geometrically expressed, for the student who has learnt "Linear Algebra". The student who knows the same material as "Matrix Theory" will need to read it more carefully, as the style of argument will be less familiar. (S)he will be well advised to do a proportion of the exercises, to consolidate understanding on matters like "how matrices multiply" which we assume familiar from *some* point of view. The next three chapters develop affine and linear geometry, with material new to most students and so more slowly taken. Chapter V sets up the algebra of tensors, handling both ends and the middle of the communication gap that made 874 U.S. "active research physicists" [Miller] rank "tensor analysis" ninth among all Math courses needed for physics Ph.D. students, more than 80% considering it necessary, while "multilinear algebra" is not among the first 25, less than 20% in each specialisation reommending it. "Multilinear algebra" is just the algebra of the manipulations, differentiation excepted, that make up "tensor analysis".

Chapter VI covers those facts about continuity, compactness and so on needed for precise argument later; we resisted the temptation to write a topology text. Chapter VII treats differential calculus "in several variables", namely between affine spaces. The affine setting makes the "local linear approximation" character of the derivative much more perspicuous than does a use of vector spaces only, which permit much more ambiguity as to "where vectors are". This advantage is increased when we go on to construct manifolds; modelling them on affine spaces gives an unusually neat and geometric construction of the tangent bundle and its own manifold structure. These once set up, we treat the key facts about vector fields, previously met as "first order differential equations" by many readers. To keep the book selfcontained we show the existence and smoothness of flows for vector fields (solutions to equations) in an Appendix, by a recent, simple and attractively geometric proof due to Sotomayor. The mathematical sophistication called for is greater than for the body of the book, but so is that which makes a student want a proof of this result.

Chapter VIII begins differential geometry proper with the theory of connections, and their several interrelated geometric interpretations. The "rolling tangent planes without slipping" picture allows us to "see" the connection between tangent spaces along a curve in an ordinary embedded surface, while the intrinsic geometry of the tangent bundle formulation gives a tool both mathematically simpler in the end, and more appropriate to physics.

Chapter IX discusses geodesics both locally and variationally, and examines some special features of indefinite metric geometry (such as geodesics *never* "the shortest distance between two points"). Geodesics provide the key to analysis of a wealth of illuminating examples.

In Chapter X the Riemann curvature tensor is introduced as a measure of the failure of a manifold-with-connection to have locally the flat geometry of an affine space. We explore its geometry, and that of the related objects (scalar curvature, Ricci tensor, etc.) important in mathematics and physics.

Chapter XI is concerned chiefly with a geometric treatment of how matter and its motion must be described, once the Newtonian separation of space and time dissolves into one absolute spacetime. It concludes with an explanation of the geometric incompatibility of gravitation with any simple flat view of spacetime, so leading on to general relativity.

Chapter XII uses all of the geometry (and many of the examples) previously set up, to make the interaction of matter and spacetime something like a visual experience. After introducing the equivalence principle and Einstein's equation, and discussing their cosmic implications, we derive the Schwarzschild solution and consider planetary motion. By this point we are equipped both to *compute* physical quantities like orbital periods and the famous advance of the perihelion of Mercury, and to *see* that the paths of the planets (which to the flat or Riemannian intuition have little in common with straight lines) correspond indeed to geodesics.

Space did not permit the coherent inclusion of differential forms and integration. Their use in geometry involves connection and curvature forms with values not in the real numbers but in the Lie algebra of the appropriate Lie group. A second volume will treat these topics and develop the clear exposition of the tensor geometric tools of solid state physics, which has suffered worse than most subjects from index debauchery.

The only feature in which this book is richer than in pictures (to strengthen geometric insight) is exercises (to strengthen detailed comprehension). Many of the longer and more intricate proofs have been broken down into carefully programmed exercises. To work through a proof in this way teaches the mind, while a displayed page of calculation merely blunts the eye.

Thus, the exercises are an integral part of the text. The reader need not *do* them all, perhaps not even many, but should *read* them at least as carefully as the main text, and think hard about any that seem difficult. If the "really hard" proportion seems to grow, reread the recent parts of the text – doing more exercises.

We are grateful to various sources of support during the writing of this book: Poston to the Instituto de Matemática Pura e Aplicada in Rio de Janeiro, Montroll's "Institute for Fundamental Studies" in Rochester, N.Y., the University of Oporto, and at Battelle Geneva to the Fonds National Suisse de la Recherche Scientifique (Grant no. 2.461–0.75) and to Battelle Institute, Ohio (Grant no. 333–207); Dodson to the University of Lancaster and (1976–77) the International Centre for Theoretical Physics for hospitality during a European Science Exchange Programme Fellowship sabbatical year. We learned from conversation with too many people to begin to list. Each author, as usual, is convinced that any remaining errors are the responsibility of the other, but errors in the diagrams are due to the draughtsman, Poston, alone.

Finally, admiration, gratitude and sympathy are due Sylvia Brennan for the vast job well done of preparing camera ready copy in Lancaster with the authors in two other countries.

Kit Dodson
ICTP, Trieste

Tim Poston
Battelle, Geneva

Contents

0. Fundamental Not(at)ions

> **"Therefore is the name of it called Babel;**
> **because the Lord did there confound the language**
> **of all the earth",**
>
> **Genesis 11, 9**

Please at least skim through this chapter; if a mathematician, your habits are probably different somewhere (maybe f^{-1} not $f^{\leftarrow}$) and if a physicist, perhaps almost everywhere.

1. Sets

A *set*, or *class*, or *family* is a collection of things, called *members*, *elements*, or *points* of it. Brackets like $\{\ \}$ will always denote a set, with the elements either listed between them (as, $\{1,3,1,2\}$, the set whose elements are the number 1, 2 and 3 – repetition, and order, make no difference) or specified by a rule, in the form $\{\, x \mid x \text{ is an integer, } x^2 = 1 \,\}$ or $\{\, \text{Integer } x \mid x^2 = 1 \,\}$, which are abbreviations of "the set of all those things x such that x is an integer and $x^2 = 1$" which is exactly the set $\{1,-1\}$. Read the vertical line $|$ as "such that" when it appears in a specification of a set by a rule.

Sets can be collections of numbers (as above), of people ({Henry Crun, Peter Kropotkin, Balthazar Vorster}), of sets ({ {Major Bludnok, Oberon}, $\{1,-1\}$, {this book} }), or of things with little in common beyond their declared membership of the set ({passive resistance, the set of all wigs, 3, Isaac Newton}) though this is uncommon in everyday mathematics.

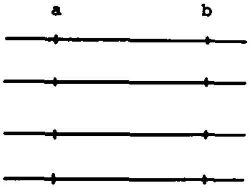
We abbreviate "x is a member of the set S" to "x is *in* S" or $x \in S$, and "x is not in S" to $x \notin S$. (Thus for instance if $S = \{1,3,1,2,2\}$ then $x \in S$ means that x is the number 1, or 2, or 3.) If x, y and z are all members of S, we write briefly $x, y, z \in S$. A *singleton* set contains just one element.

If every $x \in S$ is also in another set T, we write $S \subseteq T$, and say S is a *subset* of T. This includes the possibility that $S = T$; that is when $T \subseteq S$ as well as $S \subseteq T$.

Some sets have special standard symbols. The set of all *natural*, or "counting", numbers like $1, 2, 3, \ldots, 666, \ldots$ etc. is always N (not vice versa, but when N means anything else this should be clear by context. Life is short, and the alphabet shorter.) There is no consensus whether to include 0 in N; on the grounds of its invention several millenia after the other counting numbers, and certain points of convenience, we choose not to. The set of all *real* (as opposed to complex) numbers like 1, $\sqrt{\frac{1}{2}}$, $-\pi$, 8.2736 etc. is called $\mathbf{R}$. The *empty* set $\emptyset$ by definition has no members; thus if $S = \{\, x \in \mathbf{N} \mid x^2 = -1 \,\}$ then $S = \emptyset$. Note that $\emptyset \subseteq \mathbf{N} \subseteq \mathbf{R}$. ($\emptyset$ is a subset of any other set: for "$\emptyset \not\subseteq \mathbf{N}$" would mean "there is an $x \in \emptyset$ which is not a natural member". This is false, as there is *no* $x \in \emptyset$ which is, or is not, *anything*: hence $\emptyset \subseteq \mathbf{N}$.) Various other subsets of $\mathbf{R}$ have special symbols. We agree as usual that among real numbers

$a < b$ means "a is strictly less than b" or "$b - a$ is not zero or negative"
$a \leq b$ means "a is less than or equal to b" or "$b - a$ is not negative"

(note that for any $a \in \mathbf{R}$, $a \leq a$). Then we define the *intervals*



$[a,b] = \{\, x \in \mathbf{R} \mid a \leq x \leq b \,\}$ including ends
$]a,b[= \{\, x \in \mathbf{R} \mid a < x < b \,\}$ *not* including ends
$[a,b[= \{\, x \in \mathbf{R} \mid a \leq x < b \,\}$
$]a,b] = \{\, x \in \mathbf{R} \mid a < x \leq b \,\}$ } including *one* end.

When $b < a$, the definitions imply that all of these sets equal $\emptyset$; if $a = b$, then $[a,b] = \{a\} = \{b\}$ and the rest are empty. By convention the *half-unbounded* intervals are written similarly: if $a, b \in \mathbf{R}$ then

$$]{-\infty}, b] = \{\, x \mid x \leq b \,\},\quad [a, \infty[= \{\, x \mid x \geq a \,\},$$
$$]{-\infty}, b[= \{\, x \mid x < b \,\},\quad]a, \infty[= \{\, x \mid x > a \,\}$$

by definition, without thereby allowing $-\infty$ or ∞ as "numbers". We also call $\mathbf{R}$ itself an interval. (We may define the term *interval* itself either by gathering together the above definitions of all particular cases or – anticipating Chapter III – as a convex subset of $\mathbf{R}$.)

By $a > b$, $a \geq b$ we mean $b < a$, $b \leq a$ respectively.

A *finite* subset $S = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbf{R}$ must have a least member, $\min S$, and a greatest, $\max S$. An infinite set may, but need not have extreme members. For example, $\min[0,1] = 0$, $\max[0,1] = 1$, but neither $\min]0,1[$ nor $\max]0,1[$ exists. For any $t \in]0,1[$, $\frac{1}{2}t < t < \frac{1}{2}(t+1)$ which gives elements of $]0,1[$ strictly less and greater than t. So t can be neither a minimum nor a maximum.

We shall be thinking of $\mathbf{R}$ far more as a *geometric* object, with its points as *positions*, than as algebraic with its elements as numbers. (These different viewpoints are represented by different names for it, as the *real line* or the *real number system* or *field*.) Its geometry, which we partly explore in VII.§4, has more subtlety than high school treatments lead one to realise.

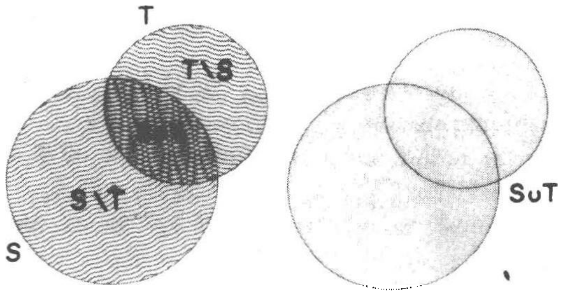


Fig. 1.1

If S and T are any two sets their *intersection* is the set (Fig. 1.1a)

$$S \cap T = \{\, x \in S \mid x \in T \,\}$$

and their *union* is (Fig. 1.1b)

$$S \cup T = \{\, x \mid x \in S, \text{ or } x \in T, \text{ or both} \,\} .$$

By S *less* T we mean the set (Fig. 1.1a)

$$S \setminus T = \{\, x \in S \mid x \notin T \,\} .$$

If we have an *indexing set* K such as $\{1,2,3,4\}$ or $\{3, \text{Fred}, \text{Jam}\}$ labelling sets $S_3, S_{\text{Fred}}, S_{\text{Jam}}$ (one for each $k \in K$) we denote the resulting set of sets $\{S_3, S_{\text{Fred}}, S_{\text{Jam}}\}$ by $\{S_k\}_{k \in K}$. K may well be infinite (for instance $K = \mathbf{N}$ or $K = \mathbf{R}$). The *union* of all of the S_k is

$$\bigcup_{k \in K} S_k = \{\, x \mid x \in S_k \text{ for } \textit{some } k \in K \,\}$$

and their *intersection* is

$$\bigcap_{k \in K} S_k = \{\, x \mid x \in S_k \text{ for } \textit{all } k \in K \,\} ,$$

which obviously reduce to the previous definitions when k has exactly two members.

To abbreviate expressions like those above, we sometimes write "for all" as $\forall$, "there exists" as $\exists$, and abbreviate "such that" to "s.t.". Then

$$\bigcap_{k \in K} S_k = \{\, x \mid x \in S_k \ \forall k \in K \,\} , \qquad \bigcup_{k \in K} S_k = \{\, x \mid \exists k \in K \text{ s.t. } x \in S_k \,\} .$$