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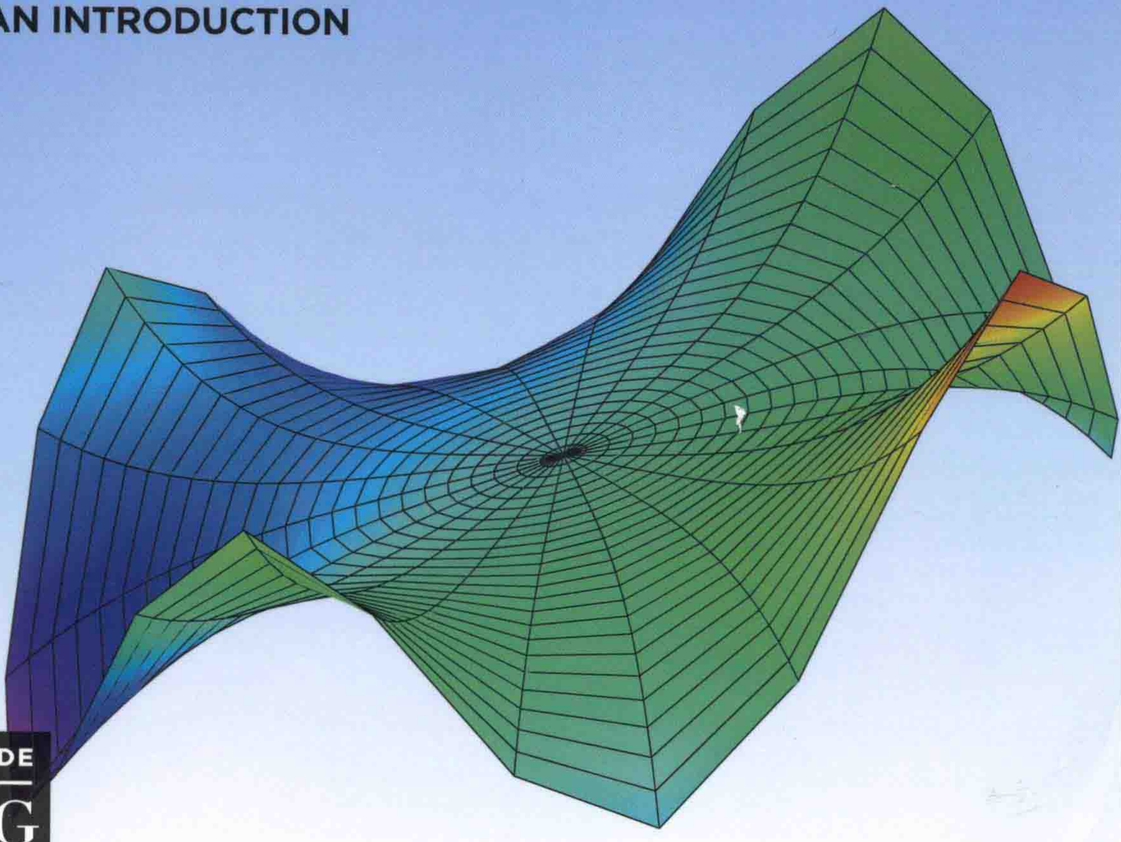
TEXTBOOK

Radu Precup

LINEAR AND SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS

AN INTRODUCTION



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Precup · Linear and Semilinear Partial Differential Equations

Non multa, sed multum

Preface

- What would be the best textbook for a brief, rapid, first and core introduction into partial differential equations (PDEs)?
- How to select and organize materials for a new textbook in PDEs, to offer not just first elements in this complex field, but also some opening toward further study, theoretical research, mathematical modeling, and applications?

The first question is explicitly or just implicitly put by oneself, by any young person interested or just constrained to take contact with PDEs.

The second question should be addressed to oneself by any author interested to produce a new introductory course in PDEs.

Answering to any of the two above questions is a very difficult task. There are several excellent texts in PDEs, each of them with its own balance of classic-modern, elementary-advanced, and theoretic-applicability. Here are some of them: Barbu [2], Bers–John–Schechter [4], Brezis [5], Courant–Hilbert [8], DiBenedetto [10], Egorov–Shubin [11], Evans [12], Folland [13], Friedman [14], Garabedian [15], John [19], Jost [20], Logan [25], Mihlin [26], Mikhailov [27], Mizohata [28], Nirenberg [32], Petrovsky [35], Rauch [41], Schwartz [45], Shimakura [46], Sobolev [49], Tikhonov–Samarskii [53], and Vladimirov [54].

Writing this book, I had in mind the above questions. The result is a book in three parts which is intended to conform to the Latin phrase "Non multa, sed multum" ("not many, but much", "not quantity, but quality"). In Part I, the reader finds an accessible, elementary introduction to linear PDEs, in the framework of classical analysis, without using the notion of distribution. However, I have considered useful in introducing, in this first part, the notion of generalized or weak solution of a boundary value problem. Weak solutions are sought in larger spaces than the common spaces of continuously differentiable functions, which are here introduced more naturally, by the completion with respect to the corresponding energetic norms. Part I can be used for a first standard one-semester course in PDEs for mathematics students.

Part II addresses to students who follow a second partial differential equations course. Here, distributions and many more results on Sobolev spaces (some of them with laborious proofs, optional for a first reading) are presented and used for

$L^p(0,T;X)$	Space of measurable functions $u:[0,T]\to X$ with $\lvert u\rvert_{L^p(0,T;X)} := \left(\int_0^T \lvert u(t)\rvert_X^p\,dt\right)^{1/p} < \infty$
$\mathcal{D}(\Omega)$	$= C_0^\infty(\Omega)$, the space of infinitely differentiable functions on Ω with compact support included in Ω
$\mathcal{E}(\Omega)$	$= C^\infty(\Omega)$
$\mathcal{D}'(\Omega)$	Space of all distributions on Ω (dual of $\mathcal{D}(\Omega)$)
$\mathcal{E}'(\Omega)$	Space of distributions with compact support
$\mathcal{S}'$	Space of temperated distributions
$L^1_{\mathrm{loc}}(\Omega)$	Space of measurable functions u on Ω with $u \in L^1(\Omega')$ for every bounded open Ω' with $\overline{\Omega'} \subset \Omega$
$H^m(\Omega)$	$= \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega)$ for all $\lvert\alpha\rvert \le m\}$ $(u,v)_{H^m} = \sum_{\lvert\alpha\rvert\le m} (D^\alpha u, D^\alpha v)_{L^2}$; $\lvert u\rvert_{H^m} = (u,u)_{H^m}^{1/2}$
$H_0^m(\Omega)$	Closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$
$H^{-m}(\Omega)$	Dual of $H_0^m(\Omega)$

Contents

Part I

Classical Theory

Chapter 1

Preliminaries

1.1 Basic Differential Operators

(a) Partial derivative operator D^α. Let $\Omega \subset \mathbf{R}^n$ be an open set and let $\alpha \in \mathbf{N}^n$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be a *multi-index.* We denote $|\alpha| = \sum_{j=1}^{n} \alpha_j$ and we define the operator

$$D^\alpha : C^{|\alpha|}(\Omega) \to C(\Omega), \quad D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}},$$

which associates to function u its partial derivative of order $|\alpha|$, of α_1 times with respect to x_1, α_2 times with respect to x_2, ..., α_n times with respect to x_n.

For example, if $\alpha_j = 0$ for $j \neq k$ and $\alpha_k = 1$, then $D^\alpha = \frac{\partial}{\partial x_k}$. Also, if $\alpha_j = 0$ for $j \neq k$ and $\alpha_k = 2$, then $D^\alpha = \frac{\partial^2}{\partial x_k^2}$.

In general, D^α can be obtained by the composition of the operators $\frac{\partial}{\partial x_j}$, $j = 1, 2, ..., n$, namely

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} ... \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Equation

$$D^\alpha u = f$$

is the simplest partial differential equation of order $|\alpha|$.

(b) Gradient.

$$\nabla : C^1(\Omega) \to C(\Omega; \mathbf{R}^n), \quad \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}\right).$$

(c) Divergence.

$$\operatorname{div} : C^1(\Omega; \mathbf{R}^n) \to C(\Omega), \quad \operatorname{div} \mathbf{v} = \sum_{j=1}^{n} \frac{\partial \mathbf{v}_j}{\partial x_j},$$

where $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n)$.

(d) Laplace's operator (Laplacian). Laplace's operator, or the Laplacian is defined as divergence of the gradient, i.e.

$$\Delta : C^2(\Omega) \to C(\Omega), \quad \Delta u = \operatorname{div} \nabla u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

Equation

$$\Delta u = f$$

is called *Poisson's equation,* and its particular case

$$\Delta u = 0$$

is named *Laplace's equation.* Any solution $u \in C^2(\Omega)$ of Laplace's equation is said to be an *harmonic function* in Ω.

(e) Directional derivative operator. Let $\upsilon \in \mathbf{R}^n$, $|\upsilon| = 1$ be a unit vector (versor of one direction in $\mathbf{R}^n$). We define the directional derivative operator as

$$\frac{\partial}{\partial \upsilon} : C^1(\Omega) \to C(\Omega), \quad \frac{\partial u}{\partial \upsilon}(x) = (\nabla u(x), \upsilon).$$

It is easy to show that

$$\frac{\partial u}{\partial \upsilon}(x) = \lim_{t \to 0+} \frac{u(x + t\upsilon) - u(x)}{t} \quad \text{for every } x \in \Omega.$$

The number $\frac{\partial u}{\partial \upsilon}(x)$ is called the *derivative of* u *in the direction* υ at x.

An important role is played by the so-called *normal derivative*, that is, derivative in the direction normal (i.e. orthogonal) to the hypersurface $\partial\Omega$, the boundary of Ω. To speak about the normal to $\partial\Omega$ at one of its points, it is necessary that Ω be "smooth" in the sense that we are going to precise.

Let $k \in \mathbf{N} \setminus \{0\}$. We say that an open set $\Omega \subset \mathbf{R}^n$ is of *class* C^k if for each point $x_0 \in \partial\Omega$, there exists $r > 0$ and a function $\varphi \in C^k(B_r(x_0); \mathbf{R})$ such that

$$\nabla\varphi(x) \neq 0 \quad \text{for every } x \in B_r(x_0),$$

$$\Omega \cap B_r(x_0) = \{x \in B_r(x_0) : \ \varphi(x) < 0\},$$

$$(\mathbf{R}^n \setminus \overline{\Omega}) \cap B_r(x_0) = \{x \in B_r(x_0) : \ \varphi(x) > 0\}.$$

The set Ω is of class C^∞, if it is of class C^k for every $k \in \mathbf{N} \setminus \{0\}$.

If Ω is of class C^1, then the vector

$$\nu(x) := \frac{1}{|\nabla\varphi(x)|} \nabla\varphi(x)$$