

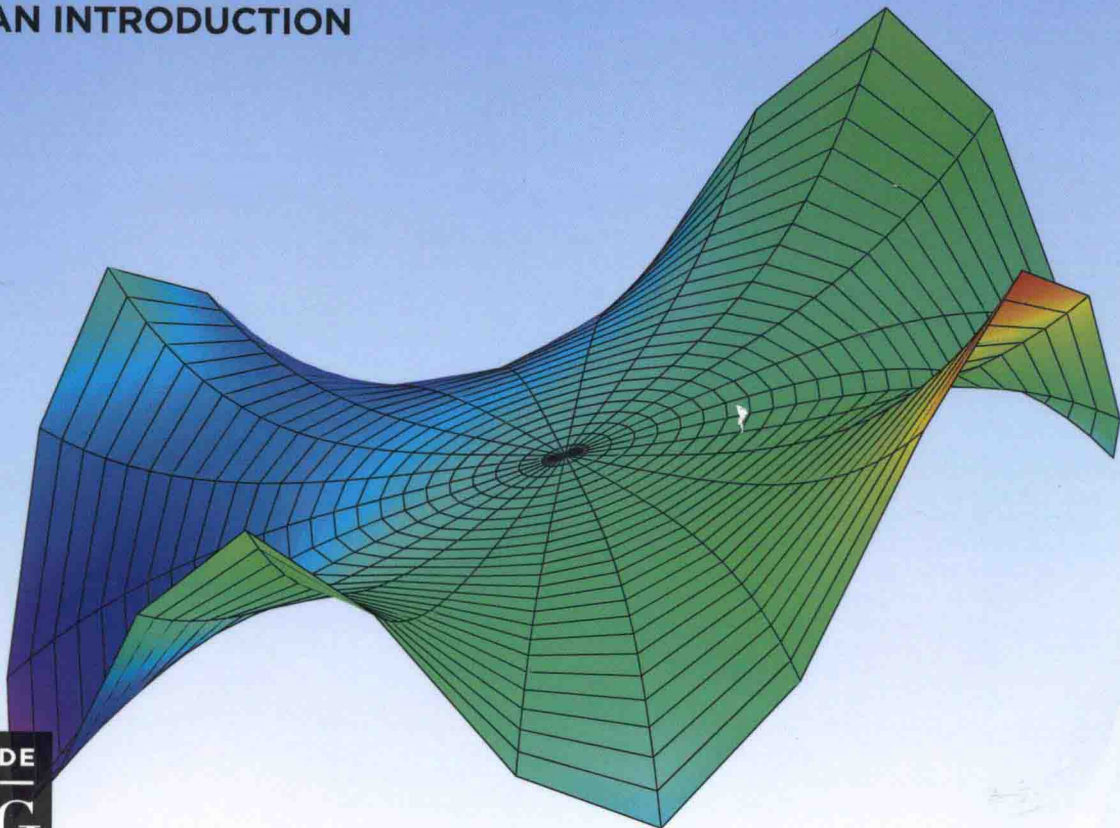
DE GRUYTER

TEXTBOOK

*Radu Precup*

# LINEAR AND SEMILINEAR PARTIAL DIFFERENTIAL EQUATIONS

AN INTRODUCTION

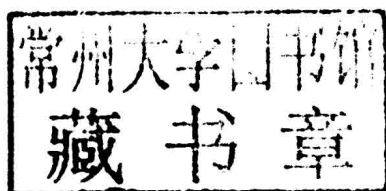


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Radu Precup

# Linear and Semilinear Partial Differential Equations

An Introduction



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Precup · Linear and Semilinear Partial Differential Equations



***Non multa, sed multum***



# Preface

- What would be the best textbook for a brief, rapid, first and core introduction into partial differential equations (PDEs)?
- How to select and organize materials for a new textbook in PDEs, to offer not just first elements in this complex field, but also some opening toward further study, theoretical research, mathematical modeling, and applications?

The first question is explicitly or just implicitly put by oneself, by any young person interested or just constrained to take contact with PDEs.

The second question should be addressed to oneself by any author interested to produce a new introductory course in PDEs.

Answering to any of the two above questions is a very difficult task. There are several excellent texts in PDEs, each of them with its own balance of classic-modern, elementary-advanced, and theoretic-applicability. Here are some of them: Barbu [2], Bers–John–Schechter [4], Brezis [5], Courant–Hilbert [8], DiBenedetto [10], Egorov–Shubin [11], Evans [12], Folland [13], Friedman [14], Garabedian [15], John [19], Jost [20], Logan [25], Mihlin [26], Mikhailov [27], Mizohata [28], Nirenberg [32], Petrovsky [35], Rauch [41], Schwartz [45], Shimakura [46], Sobolev [49], Tikhonov–Samarskii [53], and Vladimirov [54].

Writing this book, I had in mind the above questions. The result is a book in three parts which is intended to conform to the Latin phrase “Non multa, sed multum” (“not many, but much”, “not quantity, but quality”). In Part I, the reader finds an accessible, elementary introduction to linear PDEs, in the framework of classical analysis, without using the notion of distribution. However, I have considered useful in introducing, in this first part, the notion of generalized or weak solution of a boundary value problem. Weak solutions are sought in larger spaces than the common spaces of continuously differentiable functions, which are here introduced more naturally, by the completion with respect to the corresponding energetic norms. Part I can be used for a first standard one-semester course in PDEs for mathematics students.

Part II addresses to students who follow a second partial differential equations course. Here, distributions and many more results on Sobolev spaces (some of them with laborious proofs, optional for a first reading) are presented and used for



$L^p(0, T; X)$	Space of measurable functions $u : [0, T] \rightarrow X$ with $ u _{L^p(0, T; X)} := \left( \int_0^T  u(t) _X^p dt \right)^{1/p} < \infty$
$\mathcal{D}(\Omega)$	$= C_0^\infty(\Omega)$ , the space of infinitely differentiable functions on $\Omega$ with compact support included in $\Omega$
$\mathcal{E}(\Omega)$	$= C^\infty(\Omega)$
$\mathcal{D}'(\Omega)$	Space of all distributions on $\Omega$ (dual of $\mathcal{D}(\Omega)$ )
$\mathcal{E}'(\Omega)$	Space of distributions with compact support
$\mathcal{S}'$	Space of tempered distributions
$L^1_{\text{loc}}(\Omega)$	Space of measurable functions $u$ on $\Omega$ with $u \in L^1(\Omega')$ for every bounded open $\Omega'$ with $\overline{\Omega'} \subset \Omega$
$H^m(\Omega)$	$= \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \text{ for all }  \alpha  \leq m\}$ $(u, v)_{H^m} = \sum_{ \alpha  \leq m} (D^\alpha u, D^\alpha v)_{L^2}$ ; $ u _{H^m} = (u, u)_{H^m}^{1/2}$
$H^m_0(\Omega)$	Closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$
$H^{-m}(\Omega)$	Dual of $H^m_0(\Omega)$

# Contents

Preface	vii
Notation	ix
<b>I Classical Theory</b>	
<b>1 Preliminaries</b>	3
1.1 Basic Differential Operators	3
1.2 Linear and Quasilinear Partial Differential Equations	5
1.3 Solutions of Some Particular Equations	8
1.4 Boundary Value Problems	10
1.4.1 Boundary Value Problems for Poisson's Equation	10
1.4.2 Boundary Value Problems for the Heat Equation	11
1.4.3 Boundary Value Problems for the Wave Equation	12
<b>2 Partial Differential Equations and Mathematical Modeling</b>	13
2.1 Conservation Laws: Continuity Equations	13
2.2 Reaction-Diffusion Systems	16
2.3 The One-Dimensional Wave Equation	17
2.4 Other Equations in Mathematical Physics	18
<b>3 Elliptic Boundary Value Problems</b>	21
3.1 Green's Formulas	21
3.2 The Fundamental Solution of Laplace's Equation	22
3.3 Mean Value Theorems for Harmonic Functions	25
3.4 The Maximum Principle	26
3.5 Uniqueness and Continuous Dependence on Data for the Dirichlet Problem	29
3.6 Green's Function of the Dirichlet Problem	30
3.7 Poisson's Formula	31
3.8 Dirichlet's Principle	34

3.9	The Generalized Solution of the Dirichlet Problem	37
3.10	Abstract Fourier Series	42
3.11	The Eigenvalues and Eigenfunctions of the Dirichlet Problem	45
3.12	The Case of Elliptic Equations in Divergence Form	50
3.13	The Generalized Solution of the Neumann Problem	51
3.14	Complements	55
3.14.1	Harnack's Inequality	55
3.14.2	Hopf's Maximum Principle	57
3.14.3	The Newtonian Potential	59
3.14.4	Perron's Method	62
3.14.5	Layer Potentials	68
3.14.6	Fredholm's Method of Integral Equations	70
3.15	Problems	71
<b>4</b>	<b>Mixed Problems for Evolution Equations</b>	<b>87</b>
4.1	The Maximum Principle for the Heat Equation	87
4.2	Vector-Valued Functions	90
4.3	The Cauchy–Dirichlet Problem for the Heat Equation	91
4.4	The Cauchy–Dirichlet Problem for the Wave Equation	99
4.5	Problems	102
<b>5</b>	<b>The Cauchy Problem for Evolution Equations</b>	<b>109</b>
5.1	The Fourier Transform	109
5.1.1	The Fourier Transform on $L^1(\mathbf{R}^n)$	109
5.1.2	Fourier Transform and Convolution	110
5.1.3	The Fourier Transform on the Schwartz Space $\mathcal{S}(\mathbf{R}^n)$	112
5.2	The Cauchy Problem for the Heat Equation	116
5.3	The Cauchy Problem for the Wave Equation	119
5.4	Nonhomogeneous Equations: Duhamel's Principle	123
5.5	Problems	125
<b>II</b>	<b>Modern Theory</b>	
<b>6</b>	<b>Distributions</b>	<b>131</b>
6.1	The Fundamental Spaces of the Theory of Distributions	131
6.2	Distributions: Examples; Operations with Distributions	133

6.2.1	Regular Distributions	133
6.2.2	The Dirac Distribution	134
6.2.3	Differentiation	134
6.2.4	Multiplication by a Smooth Function	136
6.2.5	Composition with a Smooth Function	137
6.2.6	Convolution	137
6.2.7	Distributions of Compact Support	139
6.2.8	Weyl’s Lemma	142
6.3	The Fourier Transform of Tempered Distributions	142
6.3.1	The Fourier Transform on $\mathcal{S}'(\mathbf{R}^n)$	143
6.3.2	The Fourier Transform on $L^2(\mathbf{R}^n)$	144
6.3.3	Convolution in $\mathcal{S}'$	144
6.4	Problems	145
<b>7</b>	<b>Sobolev Spaces</b>	<b>149</b>
7.1	The Sobolev Spaces $H^m(\Omega)$	149
7.2	The Extension Operator	152
7.3	The Sobolev Spaces $H_0^m(\Omega)$	156
7.4	Sobolev’s Continuous Embedding Theorem	159
7.5	Rellich–Kondrachov’s Compact Embedding Theorem	163
7.6	The Embedding of $H^m(\Omega)$ into $C(\overline{\Omega})$	165
7.7	The Sobolev Space $H^{-m}(\Omega)$	167
7.8	Fourier Series in $H^{-1}(\Omega)$	172
7.9	Generalized Solutions of the Cauchy Problems	175
<b>8</b>	<b>The Variational Theory of Elliptic Boundary Value Problems</b>	<b>180</b>
8.1	The Variational Method for the Dirichlet Problem	180
8.2	The Variational Method for the Neumann Problem	184
8.3	Maximum Principles for Weak Solutions	186
8.4	Regularity of Weak Solutions	191
8.5	Regularity of Eigenfunctions	198
8.6	Problems	201
 <b>III Semilinear Equations</b>		
<b>9</b>	<b>Semilinear Elliptic Problems</b>	<b>208</b>
9.1	The Nemytskii Superposition Operator	208

9.2	Application of Banach's Fixed Point Theorem . . . . .	211
9.3	Application of Schauder's Fixed Point Theorem . . . . .	213
9.4	Application of the Leray–Schauder Fixed Point Theorem . . . . .	215
9.5	The Monotone Iterative Method . . . . .	218
9.6	The Critical Point Method . . . . .	220
9.7	Problems . . . . .	225
<b>10</b>	<b>The Semilinear Heat Equation</b> . . . . .	<b>227</b>
10.1	The Nonhomogeneous Heat Equation in $H^{-1}(\Omega)$ . . . . .	227
10.2	Regularity Results . . . . .	233
10.3	Application of Banach's Fixed Point Theorem . . . . .	238
10.4	Application of Schauder's Fixed Point Theorem . . . . .	241
10.5	Application of the Leray–Schauder Fixed Point Theorem . . . . .	245
<b>11</b>	<b>The Semilinear Wave Equation</b> . . . . .	<b>248</b>
11.1	The Nonhomogeneous Wave Equation in $H^{-1}(\Omega)$ . . . . .	248
11.2	Application of Banach's Fixed Point Theorem . . . . .	252
11.3	Application of the Leray–Schauder Fixed Point Theorem . . . . .	257
<b>12</b>	<b>Semilinear Schrödinger Equations</b> . . . . .	<b>262</b>
12.1	The Nonhomogeneous Schrödinger Equation . . . . .	262
12.2	Properties of the Schrödinger Solution Operator . . . . .	266
12.3	Applications of Banach's Fixed Point Theorem . . . . .	268
12.4	Applications of Schauder's Fixed Point Theorem . . . . .	272
	Bibliography . . . . .	275
	Index . . . . .	278

## **Part I**

# **Classical Theory**



# Chapter 1

## Preliminaries

### 1.1 Basic Differential Operators

(a) **Partial derivative operator**  $D^\alpha$ . Let  $\Omega \subset \mathbf{R}^n$  be an open set and let  $\alpha \in \mathbf{N}^n$ ,  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  be a *multi-index*. We denote  $|\alpha| = \sum_{j=1}^n \alpha_j$  and we define the operator

$$D^\alpha : C^{|\alpha|}(\Omega) \rightarrow C(\Omega), \quad D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}},$$

which associates to function  $u$  its partial derivative of order  $|\alpha|$ , of  $\alpha_1$  times with respect to  $x_1$ ,  $\alpha_2$  times with respect to  $x_2$ , ...,  $\alpha_n$  times with respect to  $x_n$ .

For example, if  $\alpha_j = 0$  for  $j \neq k$  and  $\alpha_k = 1$ , then  $D^\alpha = \frac{\partial}{\partial x_k}$ . Also, if  $\alpha_j = 0$  for  $j \neq k$  and  $\alpha_k = 2$ , then  $D^\alpha = \frac{\partial^2}{\partial x_k^2}$ .

In general,  $D^\alpha$  can be obtained by the composition of the operators  $\frac{\partial}{\partial x_j}$ ,  $j = 1, 2, \dots, n$ , namely

$$D^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \left( \frac{\partial}{\partial x_2} \right)^{\alpha_2} \dots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}.$$

Equation

$$D^\alpha u = f$$

is the simplest partial differential equation of order  $|\alpha|$ .

(b) **Gradient.**

$$\nabla : C^1(\Omega) \rightarrow C(\Omega; \mathbf{R}^n), \quad \nabla u = \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right).$$

(c) **Divergence.**

$$\operatorname{div} : C^1(\Omega; \mathbf{R}^n) \rightarrow C(\Omega), \quad \operatorname{div} \mathbf{v} = \sum_{j=1}^n \frac{\partial v_j}{\partial x_j},$$

where  $\mathbf{v} = (v_1, v_2, \dots, v_n)$ .



**(d) Laplace's operator (Laplacian).** Laplace's operator, or the Laplacian is defined as divergence of the gradient, i.e.

$$\Delta : C^2(\Omega) \rightarrow C(\Omega), \quad \Delta u = \operatorname{div} \nabla u = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}.$$

Equation

$$\Delta u = f$$

is called *Poisson's equation*, and its particular case

$$\Delta u = 0$$

is named *Laplace's equation*. Any solution  $u \in C^2(\Omega)$  of Laplace's equation is said to be an *harmonic function* in  $\Omega$ .

**(e) Directional derivative operator.** Let  $v \in \mathbf{R}^n$ ,  $|v| = 1$  be a unit vector (versor of one direction in  $\mathbf{R}^n$ ). We define the directional derivative operator as

$$\frac{\partial}{\partial v} : C^1(\Omega) \rightarrow C(\Omega), \quad \frac{\partial u}{\partial v}(x) = (\nabla u(x), v).$$

It is easy to show that

$$\frac{\partial u}{\partial v}(x) = \lim_{t \rightarrow 0^+} \frac{u(x + tv) - u(x)}{t} \quad \text{for every } x \in \Omega.$$

The number  $\frac{\partial u}{\partial v}(x)$  is called the *derivative of  $u$  in the direction  $v$  at  $x$* .

An important role is played by the so-called *normal derivative*, that is, derivative in the direction normal (i.e. orthogonal) to the hypersurface  $\partial\Omega$ , the boundary of  $\Omega$ . To speak about the normal to  $\partial\Omega$  at one of its points, it is necessary that  $\Omega$  be "smooth" in the sense that we are going to precise.

Let  $k \in \mathbf{N} \setminus \{0\}$ . We say that an open set  $\Omega \subset \mathbf{R}^n$  is of class  $C^k$  if for each point  $x_0 \in \partial\Omega$ , there exists  $r > 0$  and a function  $\varphi \in C^k(B_r(x_0); \mathbf{R})$  such that

$$\begin{aligned} \nabla\varphi(x) &\neq 0 \quad \text{for every } x \in B_r(x_0), \\ \Omega \cap B_r(x_0) &= \{x \in B_r(x_0) : \varphi(x) < 0\}, \\ (\mathbf{R}^n \setminus \overline{\Omega}) \cap B_r(x_0) &= \{x \in B_r(x_0) : \varphi(x) > 0\}. \end{aligned}$$

The set  $\Omega$  is of class  $C^\infty$ , if it is of class  $C^k$  for every  $k \in \mathbf{N} \setminus \{0\}$ .

If  $\Omega$  is of class  $C^1$ , then the vector

$$v(x) := \frac{1}{|\nabla\varphi(x)|} \nabla\varphi(x)$$