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Erdős–Ko–Rado Theorems: Algebraic Approaches

CHRISTOPHER GODSIL

KAREN MEAGHER

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CHRIS GODSIL
University of Waterloo, Ontario, Canada

KAREN MEAGHER
University of Regina, Saskatchewan, Canada

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ERDŐS–KO–RADO THEOREMS: ALGEBRAIC APPROACHES

Aimed at graduate students and researchers, this fascinating text provides a comprehensive study of the Erdős–Ko–Rado (EKR) Theorem, with a focus on algebraic methods. The authors begin by discussing well-known proofs of the EKR bound for intersecting families of sets. The natural generalization of the EKR Theorem holds for many different objects that have a notion of intersection, and the bulk of this book focuses on algebraic proofs that can be applied to these different objects. The authors introduce tools commonly used in algebraic graph theory and show how these can be used to prove versions of the EKR Theorem. Topics include association schemes, strongly regular graphs, distance-regular graphs, the Johnson scheme, the Hamming scheme, and the Grassmann scheme. The book also gives an introduction to representation theory (aimed at combinatorialists) with a focus on the symmetric group. This theory is applied to orbital schemes, concentrating on the perfect matching scheme and other partitions, and to conjugacy class schemes, with an emphasis on the symmetric group.

Readers can expand their understanding at every step with the 170 end-of-chapter exercises. The final chapter discusses in detail 14 open problems, each of which would make an interesting research project.

Chris Godsil is a professor in the Combinatorics and Optimization department at the University of Waterloo, Ontario, Canada. He authored (with Gordon Royle) the popular textbook Algebraic Graph Theory. He started the Journal of Algebraic Combinatorics in 1992 and he serves on the editorial board of a number of other journals, including the Australasian Journal of Combinatorics and the Electronic Journal of Combinatorics.

Karen Meagher is an associate professor in the Department of Mathematics and Statistics at the University of Regina, Saskatchewan, Canada. Her research area is graph theory and discrete mathematics, in which she has published around 25 journal articles.

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To our families:

Gillian, Emily and Nicholas
Paul, Ada and Dash

Preface

There are 280 partitions of $\{1, \ldots, 9\}$ into three pairwise disjoint triples. Say two such partitions α and β are skew if each triple in α contains one point from each triple in β. We can now define a graph with our 280 partitions as vertices, where two partitions are adjacent if and only if they are skew. A coclique in this graph is a set of partitions such that no two are skew. We ask the innocent question: how large can a coclique be?

There is an easy lower bound. Let Ω be the set of partitions such that the points 1 and 2 lie in the same triple. (There are 70 of these.) Clearly no two partitions in Ω are skew and so we now have a lower bound of 70 on the size of a coclique. But now we have two questions. Can we do better? And, if not, are there any cocliques of size 70 that do not have this form?

Karen asked Chris the first question in 2001, and by a process of induction and complication, we were led to this book. The complications arise because there is a close connection to the Erdős–Ko–Rado Theorem, one of the fundamental results in combinatorics. This theorem provides information about systems of intersecting sets. A family $\mathcal{F}$ of subsets of a ground set – it might as well be $\{1, \ldots, n\}$ – is intersecting if any two sets in $\mathcal{F}$ have at least one point in common. More generally it is t-intersecting if any two elements of $\mathcal{F}$ have at least t points in common. The most commonly stated form of the EKR Theorem is the following.

0.0.1 Theorem. *If $\mathcal{F}$ is an intersecting family of k-subsets of $\{1, \ldots, n\}$, then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

If equality holds then $\mathcal{F}$ consists of the k-subsets that contain a given point of the underlying set. □

In our view this theorem has two parts: a bound and a characterization of families that meet the bound.

One reason this theorem is so important is that it has many interesting extensions. To address these, we first translate it to a question in graph theory. The Kneser graph $K(n, k)$ has all k-subsets of $\{1, \ldots, n\}$ as its vertices, and two k-subsets are adjacent if they are disjoint. (We assume $n \geq 2k$ to avoid trivialities.) Then an intersecting family of k-subsets is a coclique in the Kneser graph, and we see that the EKR Theorem characterizes the cocliques of maximum size in the Kneser graph. So we can seek to extend the EKR Theorem by replacing the Kneser graphs by other interesting families of graphs. The partition graphs just discussed provide an example.

There is a second class of extensions of the EKR Theorem. In their famous 1961 paper Erdős, Ko and Rado proved the following:

0.0.2 Theorem. *If $\mathcal{F}$ is a t-intersecting family of k-subsets of $\{1, \ldots, n\}$ and n is large enough, then*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

If equality holds then $\mathcal{F}$ consists of the k-subsets that contain a given t-subset of the underlying set. □

There are graph-theoretic analogs of this question too, but we have to work a little harder. In place of the Kneser graphs, we use the Johnson graphs $J(n, k)$. The vertices of $J(n, k)$ are the k-subsets of $\{1, \ldots, n\}$, but now we deem two k-subsets to be adjacent if they have exactly $k - 1$ points in common. Again we assume $n \geq 2k$. We can show by induction that $J(n, k)$ has diameter k and thus two k-subsets are adjacent in $K(n, k)$ if and only if they are at maximum possible distance in $J(n, k)$.

Define the width of a subset of the vertices of a graph to be the maximum distance between two vertices in the subset. Then our first version of the EKR Theorem characterizes the subsets of maximum size in $J(n, k)$ of width $k - 1$, and the second version the subsets of maximum size with width $k - t$. All known analogs of the EKR Theorem for t-intersecting sets can be stated naturally as characterizations of subsets of width $d - t$ in a graph of diameter d. However, such theorems have only been proved in cases where the distance graphs form an association scheme (which for now means that they fit together in a particularly nice way). To give one example, we can replace k-subsets of

$\{1, \ldots, n\}$ by subspaces of dimension k over a vector space of dimension n over $GF(q)$.

What of this book? One goal has been to show how the EKR Theorem can be tackled using tools from algebraic graph theory. But we are not zealots, and we begin by discussing most of the known proofs of the EKR bound for intersecting families; these are not algebraic. We go to develop many of the tools we need, and then we apply them to strongly regular graphs. We develop the basic properties of the Johnson scheme, and using these we offer two proofs of the EKR Theorem for intersecting families of k-sets.

We present a version of Wilson's proof of the EKR Theorem for t-intersecting families of k-subsets when $n \geq (t+1)(k-t+1)$. The main novelty is that in order to derive the characterization of the maximum families, we make explicit use of the concepts of width and dual width. (Here we are following important work by Tanaka [160].) It is comparatively easy to extend this approach to t-intersecting families of k-dimensional subspaces of a vector space over a finite field. We complete the first half of the book by treating EKR problems on words; here the Hamming schemes provide a natural framework.

In the second part of the book, we consider versions of EKR on sets of permutations and partitions. For this we need to make use of the fact that in these problems there is a natural action of the symmetric group, and this means we need information about the representation theory of the symmetric group. We treat this in some detail (although in many cases we refer the reader to the literature for proofs).

In the original version of the EKR Theorem there is a requirement that n be large relative to the size of the subsets and the size of the intersection; this condition cannot be dropped, and a significant body of work was required to determine the exact bound on n. For many of the analogs of the EKR Theorem a bound, analogous to this lower bound on n, is required. It may be possible to obtain strong results and the assumption that n (or its analog) is "sufficiently large." Such results are not the focus of this book; we are more interested in the combinatorial details involved in finding exact results.

We assume a working knowledge of graph theory, but otherwise we have tried to keep things self-contained. There are exercises of varying difficulty at the end of each chapter. In general, if there is a reference attached to the exercise, expect it to be more challenging.

This book is the culmination of many years of work, and there are many people whom we wish to thank for their assistance and encouragement in writing this book as well as many interesting and illuminating discussions. We specifically wish to thank following people: Bahman Ahmadi, Robert Bailey, Adam Dyck, Ferdinand Ihringer, Bill Martin, Mike Newman, Alison Purdy and Pablo Spiga. Finally, we offer our gratitude to Brendan Rooney, who completed a detailed edit of the entire book. He corrected many errors, both mathematical and grammatical, and his contribution has significantly improved this book!

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