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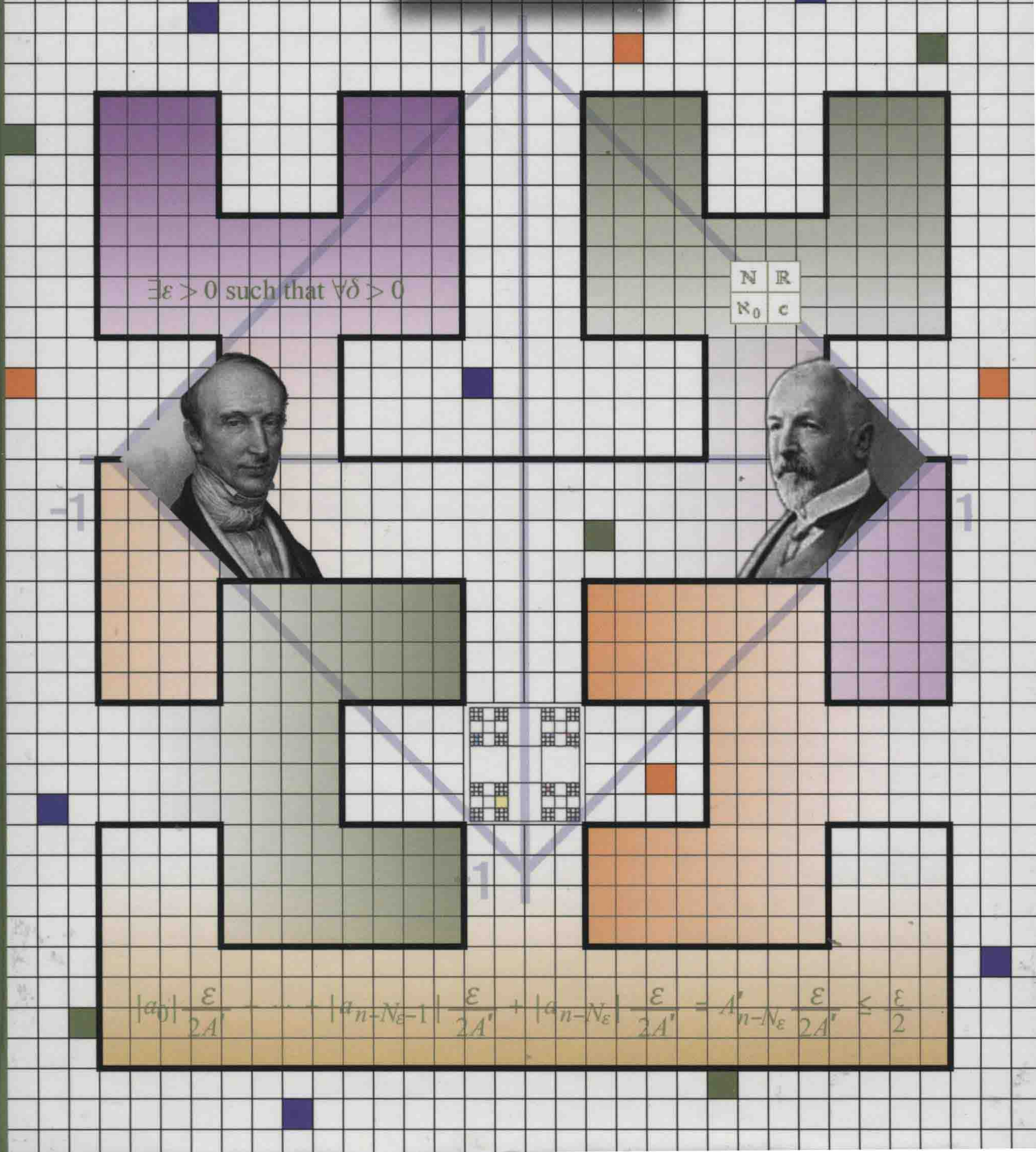
An Invitation to Real Analysis

LUIS F. MORENO



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To my wife, patient donor of many hours of quiet time, to my parents, and ultimately, to Jesus of Nazareth, Son of the most high God.

Call unto Me, and I will answer thee, and shew thee great
and mighty things, which thou knowest not.
Jeremiah 33:3

To the Student

We are all students, or pupils (*mathetés* in Greek), and should never cease to be so. This book is an invitation and an introduction to real analysis, a branch of mathematics that most students of calculus find to be quite unexpected. Broadly speaking, real analysis studies the foundational principles of the real number system, and the properties of real functions and series, often with regard to limits. As such, one of its accomplishments was to create the theoretical foundations for calculus. Thus, some of the theorems that were—do we dare say—old friends from Calculus I and II, are presented in a more general setting, and always with proofs. But there will be many new and brilliant theorems about sets, functions, series, and other topics. Also, real analysis is where mathematics finally came to terms with the infinite. This aspect has always fascinated me; I am sure that you will feel the same way. No other human endeavor studies infinity quantitatively and in so many different aspects. I invite you to enter this fascinating, meticulous subject, and meet the people who matured it.

Some comments from those who have accepted the invitation and stepped through the door are, (1) "Do we really need to prove that, isn't it pretty obvious?" Also, (2) "It's almost all about #*x@ inequalities!" (We may call "#*x@" a "variable adjective," since its meaning changes by different word substitutions.) And, (3) "Where did all the nice pictures of things go?" And sometimes, (4) "That was tough to understand, but it's awesome!" And from my memory of my first real analysis course, (5) "Who were those guys who came up with these amazing ideas?" Each of the students' comments above carries worthwhile insight.

(1) Regarding the first one, you already know that some things are obvious only because they haven't been considered very carefully. Analysis is in the business of establishing the truth of statements by deduction, not by "hand-waving." And sometimes, an abstract mathematical claim becomes clear to you only after conquering its proof. Writing your own proof is almost an art, full of hunches, competing viewpoints, dead ends, and the like. You will learn to devise proofs that are logically sound (avoiding circular reasoning, for instance) and even elegant, but this comes only with patient practice, so that homework cannot be shortchanged.

In this book, to show (or demonstrate) that something is true means the same as to prove that it is true. Alternatively, the words "convince yourself" and "verify that" allow less rigorous, intuitive, thinking.

(2) Regarding the second comment, while most of calculus concentrates on equations, many concepts in real analysis require inequalities. Your thinking about how to preserve the size

order of things will have to be strengthened. By doing the exercises, you will learn to dissect, construct, and apply inequalities effectively.

(3) The comment about illustrations is quite interesting. Pictures may be misleading in mathematics, as you will soon see (for this reason, some advanced texts have absolutely no figures at all). Thus, although there are plenty of them in this book, the illustrations are intended to be supportive, suggestive, and schematic. How to draw figures that clarify an issue but don't assume too much is something to be learned. Thus, draw many pictures as part of your diligently done homework.

(4) With regard to difficulty of understanding, we know that some things are difficult just because they are communicated badly. But in analysis, some things are difficult to understand because they are genuinely profound, in fact so profound that mere prose cannot be used to communicate the concepts. This is why the marvelous symbolic language of mathematics is necessary. Entering real analysis, you are as a young artist who has just been handed a palette of colors never seen before. The symbols are the colors with which mathematics is painted. In order to become handy with them, you must use them often in homework exercises.

It seems that homework is the string that ties the whole bundle securely. I think this is close to the truth. Consider the following table with four combinations of ingredients. The upper headings of the table are the two answers to a question we always have about a topic, call it Topic X: "Do I understand topic X?" To keep things manageable, we won't allow tentative answers to this question. This question may be answered with certainty *after* the fact of learning, for example, "Do I know how to multiply three-digit whole numbers?" But for topics that you are in the *process* of learning ("Do I understand how to detect a cluster point?"), the answer won't be known for certain. A graded exam, for instance, may determine the answer quite clearly, but by then, it may be too late. Down the left side of the table, you must ask yourself whether you *think* that you understand Topic X. This is your subjective appraisal of your understanding of (or skill with) Topic X, at the present time. Again, limit your answer to a simple "yes" or "no." This is an honest appraisal; you are not dealing with unknowns here.

	I actually do understand Topic X	I actually don't understand Topic X
I think that I understand Topic X	I	II
I don't think that I understand Topic X	III	IV

We now consider the situations marked I, II, III, and IV. What is the effect of situation I? Here, you are confident that you understand Topic X, and, in fact, you actually do know the topic. This is the case where effective study of homework exercises, discussion with the instructor, and so forth, have made you competent in the topic. It is very likely that you will do well on a test of Topic X, and you will remember it well later on. Situation I lowers anxiety, and it is generally a good position to be in. Situation IV isn't bad either, assuming that there is time to rectify the problem. As you see, you have identified the lack of understanding, and you may get together with a couple of friends and a pot of strong coffee to "figure it out." At some imperceptible moment, you will cross over from IV to III, and soon after, you will arrive at situation I. Situation III is problematic only in that you might spend too much time grinding away at Topic X and not budget time for other necessary things.

Finally, consider situation II, where you are falsely assuming that you know Topic X. This case results in a false sense of security, which is usually catastrophic in mathematics. Perhaps Topic X was clear when you left the lecture. A few of the (as it turned out) easier homework exercises were casually done. But when the topic had to be applied in a later context, you brushed aside the details in order to get the main job done. The truth about Topic X is rudely discovered only at test time. Fortunately, effectively done homework can suddenly push you from II to IV, and then slowly but steadily to I.

A Homework Strategy

There are over 600 exercises in this book. The table above supports the idea that studying mathematics and doing homework effectively are learned behaviors. You surely know some techniques that help you grasp ideas more efficiently in any subject. This book will suggest several methods to gain understanding (to move you towards situation I), and you should try them all. For example, after creating a proof that a function is continuous at a point, an exercise may ask you to change the function slightly to see how this affects the proof. Another strategy is that of simplifying the assumptions given in an exercise, and then seeing what you can deduce. A third strategy is to switch to a different form of proof (remember proof by contradiction?). In time, strategies such as these will become part of your "solver's toolkit" for attacking problems in general.

Solutions to odd-numbered exercises appear at the end of each chapter. But how do you know that a proof written by you is "airtight?" Or, when the solution isn't listed, could your solution have an undetected fatal flaw? My suggestions to prevent such things from happening are as follows:

(1) Do as much of the exercise as you can by applying the concepts that you have just studied, along with past concepts as needed, without looking at a hint or solution. You may need to apply one of the tools in your toolkit. If you succeed in answering the exercise, then congratulations, the wrestling match was worth it! What's more, your solution or proof may be more elegant than the text's, and that is the best of all situations.

(2) Often, hints are attached to the exercises. If you can't see a first step, or can't even grasp the problem, look at the hint, or a part of the hint. Pursue any line of attack that the hint suggests. That may be all it takes to jump-start your solution. Then, compare your work with that of others in the class. Together, you may find an error of some kind. Be not discouraged, for this is a superb way to learn. Finally, compare your work with the given solution (if available).

(3) If the hint takes you only part way, or nowhere at all, you have nevertheless begun to unravel the knot. Now, look at the *first sentence* of the solution, if it is there. Notice how the hint is applied, and try to proceed to the solution.

(4) If the solution is supplied, and if step (3) didn't help completely, then look at more of the solution, but not all of it. At this point, you will see more of the path to the conclusion, but it is important for you to follow the signs in a logical manner. You may come to realize that the solution is staring right at you! If the solution is not supplied, it is time to ask about it in class. Your instructor may have the full solutions manual.

Of course, the worst thing to do is to make a dispirited attempt at the exercise, and then read the full solution. That's comparable to reading the spoiler to a suspenseful film, and, just as bad, it draws you into situation II.

The People of Mathematics

Comment number five is addressed here. Mathematics shouldn't be presented as a disembodied collection of facts. Common parlance, especially what comes out of Hollywood, has cheapened the meaning of genius. I hope the short passages included here will reinstate the proper sense of the word. Most of these men—and more recently, women—of mathematics displayed an insight into the unknown that will seem to you almost unearthly. Even now, with over a century of hindsight on their work, the vision of mathematicians such as Bernhard Riemann and Georg Cantor is still stunning, especially because the mathematical analysis they wrote about exists outside of physical, sensory existence. Nevertheless, many of these geniuses were also superb at applied mathematics, in fields such as astronomy, thermodynamics, and engineering.

The visual arts handle poorly both the work of mathematicians and the individuals themselves. The cinematic industry regularly displays its incompetence in portraying mathematicians (and scientists in general). Only recently has an occasional documentary realistically discussed the struggles and triumphs of a few mathematicians (I'm thinking of Andrew Wiles (1953–)). And so, enjoy the selections herein and look into the references under "History of Mathematics" in the bibliography for the real story.

To the Instructor

With this book I hope to ease a student's transition from what may be called a "consumer of mathematics" up through calculus II, into one beginning participate in its creative process. Specifically I mean that with your guidance, this book will help a student to

(1) be able to transform a faulty proof into a good, maybe elegant, one,
(2) correctly use symbols, and correctly apply definitions and theorems,
(3) become aware of the underlying laws of logic,
(4) understand the axiomatic development of real numbers,
(5) have a deeper understanding of the core topics of real analysis, and
(6) appreciate the brilliant minds who, in creating this subject, brought clarity and rigor to the rapidly growing bundle of mathematical techniques of the late 18th century.

A somewhat abbreviated coverage of this book would encompass Chapters 4 through 50, skipping Chapters 11 and 32. This covers (some of) the foundations of real numbers, countable and uncountable sets, sequences, series, the properties of functions, differentiation, and Riemann integration. Those chapters, most of which are short, may be covered in a 14-week semester, with six weeks covering four sections per week and eight weeks with three per week, leaving time for testing. In a 15-week semester, all the chapters can be covered, with a cursory look at Chapter 1. This would then include a more complete treatment of the axioms, an investigation of transfinite cardinals, and an introduction to metric spaces, the topology of the real line, and the Cantor Ternary set.

A diverse, annotated bibliography is broken up into four sections to encourage students to use it.

Over 600 exercises are included. A complete solutions manual is available to the instructor.

Several techniques address objective (1), the ability of students to recognize when their written proofs are valid and understandable. Some theorems have proofs with a parallel case that is left as an exercise, since the given case then acts as a scaffold for the student. Certain corollaries' proofs are also placed as exercises for the same reason. Some proofs to be done as exercises have a step-by-step form that is (I think) pedagogically effective.

Regarding objective (2), there is occasional redundancy and repetition, especially if the concept in question hasn't been used in a while. New or unfamiliar symbols are related to past concepts, and collected in an appendix. Many figures are employed. Some definitions (e.g., that of a function unbounded on an interval) are included for clarity and reference, and to reinforce goal (3). Many exercises have hints, to be used judiciously per the preface to the student.

As for objective (3), understanding valid inference is the purpose of an introduction to logic in Chapter 1. I don't just mention it and then leave it behind. The difference between necessity and sufficiency is constantly stressed. The negation of quantifiers appears in step-by-step fashion where appropriate.

Objective (4) is accomplished in a unified way by forming a pedigree of axioms, so that the student can see at a glance which axioms are inherited from the previous number set, and therefore, which one (or ones) form the discriminating factor for the next, richer, set. In this way, a student can see both the richness and the limitation of $\mathbb{Q}$, for instance.

Objective (5) is, of course, the main purpose of this book. A few slightly less traditional topics have been included, some as appendices. As examples, a subsection on quadratic extensions adds a little flesh to what is often just mentioned as "algebraic numbers." A theorem about the division of power series seems natural after the work on products of series. The full proof of Newton's binomial series theorem is left to a more advanced text; instead, the convergence behavior at the endpoints is analyzed since it requires good use of previous concepts (and it's not often found in other texts). The proof that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ is different, using the sequential criterion for convergence. Continued fractions are introduced in an appendix. Another appendix contains the full proof of l'Hospital's rule (again, something not often found). And surely, students will be intrigued by the curious Farey sequences found in another appendix.

Regarding objective (6), short notes about many mathematicians are interspersed throughout the text. I hope this helps to break the theorem-proof-example cadence which can't be sustained for long periods.

Numerous references to articles in *The College Mathematics Journal*, *Mathematics Magazine*, *The American Mathematical Monthly*, and other sources in mathematics may be fruitfully assigned to students in order to deepen their understanding of analysis and the mathematicians themselves.

Acknowledgments

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