Foundations in Modern Mathematics

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WAKE FOREST COLLEGE

Blaisdell Publishing Company

A DIVISION OF GINN AND COMPANY

Waltham, Massachusetts · Toronto · London

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 66–21104
PRINTED IN THE UNITED STATES OF AMERICA.

FOUNDATIONS IN MODERN MATHEMATICS

A BLAISDELL BOOK IN PURE AND APPLIED MATHEMATICS

To my parents

EOLINE E. MAY and GORDON H. MAY

Preface

Modern mathematics is exemplified by those methods that attempt to give precise expression to logical thought. Currently, mathematicians are developing these methods from a foundation in which the concepts of set, relation, and function serve as cornerstones. It is suitable, then, that these should be the concepts that are central to the theme of *Foundations in Modern Mathematics.*

General Purpose. This book investigates the aforesaid concepts in a setting taken from the areas of algebra and trigonometry. Throughout, the aim has been to use a *style of expression* and a *choice of material* that will:

- Convey a real purpose and a sense of value for the mathematics that the student is asked to study.
- Prepare a strong foundation for the next course in sequence—which, it is expected, will be calculus.
- Give occasional glimpses into areas that the student would explore if he pursued his training in mathematics to more advanced levels.

Style of Expression. Knowledge will usually flourish when a creative mind is associated with a stimulating environment. To encourage such an environment this book has tried to use a style of expression that will promote clarity, but will not insult the intelligence of either the student or teacher. Among the basic considerations that have helped determine the style are:

- Each section is short enough to be covered in one lecture period.
- Symbolic abbreviations are used with discretion. In general, the interpretation of a sentence does not require the reader to translate very many symbols.
- Notational devices are used only if the instructor can reproduce them easily on the blackboard.

- Concise explanations are given in preference to lengthy discussions. This keeps the book primarily as a teaching instrument and does not let it impose upon the role of the teacher.
- The material on a given page is made to look as appetizing as possible. There are no long paragraphs, and figures appear frequently.

Choice of Material. Most of the topics in this book treat either the properties of elementary functions or the algebraic structure of number systems. The arrangement and presentation of these topics was done with a twofold purpose in mind. First, that the content should unfold logically through the practice of good theory. Second, that a need for having sound theory should be promoted by methods which the student already values.

There are a number of distinctive features in this book which were devised to assist with topics that are often troublesome. For instance:

- An appreciation for the logical structure of the real number system is motivated by first working with very simple abstract models. This provides for computations involving familiar methods of algebra, but demands that the definitions of the field properties be applied.
- A natural transition is made from the trigonometric functions of angles to the trigonometric functions of numbers. This comes from letting a real number play a part in the definition of an angle.
- Each of the more difficult concepts concerning functions is given in connection with a useful application. For example, the usual algebra of functions is given as the foundation for the trigonometric identities.
- Mathematical induction is presented during the study of sequences. This allows much of the inductive method to become an application of familiar functional concepts.

Preliminary versions of this book have been class tested at Wake Forest College. This extended over two terms with J. Robert Johnson and Temple H. Fay serving as the instructors. Their findings indicate that the book's content is ample for a course that carries from three to five semester hours credit (or its equivalent in a different academic system).

Acknowledgment. Any acclaim for this book will probably be given to me, but it is due equally to my wife, Harriet. I gave her the tedious role of typist and critic. She returned this with a continuing contribution of patience and encouragement.

W. Graham May

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IN MODERN MATHEMATICS

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CHAPTER 1

Sets, Relations, and Functions

From the pioneering work of G. Cantor in the latter part of the nineteenth century, set theory has developed into an important branch of modern mathematics. Today, mathematicians find the use of sets to be virtually indispensable in supplying both precision and generality to the description of their ideas.

We shall not be concerned with a logical development of set theory. Instead, we shall examine some of the important properties and terminology of sets and use these as tools in our own mathematical investigations.

1.1 Description of Sets

The following definition should satisfy everyone's intuitive concept of what a set should be. Although it will not survive the scrutiny of formal logic, it is entirely adequate for our purpose.

▶ DEFINITION 1.1 *A* set *is a collection of objects. Each separate object of the collection is called an* element *or* member *of the set.*

Capital letters are customarily used to represent sets and small letters to represent the elements of a set.

▶ DEFINITION 1.2 *The symbol "$x \in A$" means that an object x is an element of a set A. The symbol "$x \notin A$" denotes the negation of $x \in A$; that is, that object x is not an element of set A.*

We adopt two methods of describing sets.

Method 1. When feasible, we may list the names of the elements and enclose the list within braces.

EXAMPLE 1 The symbol $\{1, 2, 3\}$ represents a set whose elements are the positive integers 1, 2, and 3.

EXAMPLE 2 The symbol $\{1, 2, 3, \ldots, 100\}$ represents a set whose elements are the positive integers 1 through 100.

EXAMPLE 3 The symbol $\{1, 2, 3, \ldots\}$ represents a set whose elements are all of the positive integers.

Examples 2 and 3 illustrate the notational device of "..." which means to continue according to the established pattern. We shall use this device only if the partial listing is adequate to make the intended pattern absolutely clear.

Remark • It is entirely a question of one's judgment as to whether or not some partial listing has established an unmistakable pattern. This fact is given further consideration during the study of sequences that appears in Chapter 7.

Method 2. We can specify a set by stating a property that is possessed by only the elements of the set in question. To be more precise, let U be a given set and suppose $S(x)$ represents some declarative statement about a thing named x. Suppose, further, that $S(x)$ is either true or else false each time x represents an element of U. In this event, $\{x \mid x \in U \text{ and } S(x)\}$ represents the set of exactly those elements of U that make $S(x)$ true.

EXAMPLE 4 Let $U = \{-3, 5, -\frac{1}{2}, \frac{3}{5}\}$. With Method 1, describe the set $\{x \mid x \in U \text{ and } S(x)\}$ where $S(x)$ is the statement:

(a) x is a positive integer.
(b) x is a negative number.
Solution • (a) $\{5\}$. (b) $\{-3, -\frac{1}{2}\}$.

The set U is often called a *universal* set since its elements are all the things which interest us in a particular discussion. The universal set of one discussion need not consist of the same elements as that of another discussion. Furthermore, the universal set will be designated by whatever letter or symbol seems to best suit the occasion.

We almost always write a statement $S(x)$ with mathematical notations, as shown in Example 5.

EXAMPLE 5 Let I be the set of all integers. The statement specifying the set $\{x \mid x \text{ is an integer whose sum with two is four}\}$ can be shortened to yield $\{x \mid x \in I \text{ and } x + 2 = 4\}$. In this case, the universal set is I. Of course, the set specified in this example can be described in the enumerative language of Method 1. To do so, we simply write $\{2\}$.

When Method 2 is employed, any one of several statements can be used to specify the same set.

EXAMPLE 6 The set whose elements are 2, 4, and 8 could be shown as

$$\{x \mid x \in I \text{ and } (x - 2)(x - 4)(x - 8) = 0\}$$

or as

$$\{x \mid x \in I \text{ and } x = 2^n \text{ for } n = 1, 2, \text{ and } 3\}.$$

Remark • Our chief use of Method 1 will be for simple demonstrations. In general applications, we are usually as interested in the idea that produces a set as we are in the elements themselves. To show the underlying ideas, we use Method 2.

In order that any meaningful statement $S(x)$ shall specify a set, it is necessary that we conceive of a set that has no elements at all.

▶ DEFINITION 1.3 *The* empty *set is a set having no elements. The symbol* "$\varnothing$" *will frequently be used to represent this set.*

EXAMPLE 7 (a) The set $\{x \mid x \text{ is a positive integer, and } x + 2 = 1\}$ is the empty set. The statement specifying this set is meaningful in that any positive integer makes it either true or else false—in this case, always false.

(b) $\{x \mid x \in I \text{ and } x \neq x\}$ is also a suitable expression for the empty set.

PROBLEMS

Describe each of the following sets by Method 1.

1. The set of all digits used in our decimal system.
2. The set of all integers which satisfy the equation $x^2 = 9$.
3. $\{x \mid 5x = 10\}$.
4. The set of all integers which are divisible by 3.

5. $\{x \mid x = 3n$, and n is any positive integer$\}$.
6. $\{x \mid (x + 2)(x - 5) = 0\}$.
7. The set of all positive integers which are greater than 7 but less than 10.
8. $\{x \mid x(x + 1) = 0\}$.
9. The set of all positive odd integers which are less than 100 and are multiples of 5.

Describe each of the following sets by Method 2.

10. $\{-2\}$.
11. The set of all integers which are not divisible by 2.
12. $\{2, 4, 6, 8, \ldots\}$.
13. The set of all integers which satisfy the equation $x^2 = -9$.
14. $\{2, 4, 6, 8, \ldots, 20\}$.
15. $\{-2, 2\}$.
16. The set of all positive integers that are multiples of 7.
17. The set of all negative integers that are multiples of 2.
18. $\{-1, -3, -5, -7, \ldots\}$.

For Problems 19–25, list the elements of the set specified by the statement $S(x)$. In each case consider the universal set to be $U = \{0, -1, 2, -5, 7, 9\}$.

19. $S(x)$: x is an integer.
20. $S(x)$: $x + 1 = 3$.
21. $S(x)$: x^2 is a positive integer.
22. $S(x)$: 9 is divisible by x.
23. $S(x)$: $x - 1 = 4$.
24. $S(x)$: $x(x + 2) = 0$.
25. $S(x)$: x is greater than 0.

Each of Problems 26–30 refers to the sets $A = \{1, -3, 2, 4\}$ and $B = \{2, 3, 4, 5\}$. In each case list the elements of the specified set.

26. $\{x \mid x$ is in at least one of the sets$\}$.
27. $\{x \mid x$ is in one and only one of the sets$\}$.
28. $\{x \mid x$ is in one but not both of the sets$\}$.
29. $\{x \mid x$ is in both of the sets$\}$.
30. $\{x \mid x$ is in one but at most one of the sets$\}$.

1.2 Subsets

There are several standard means whereby given sets will give rise to other sets. The simplest of these is through the formation of subsets.