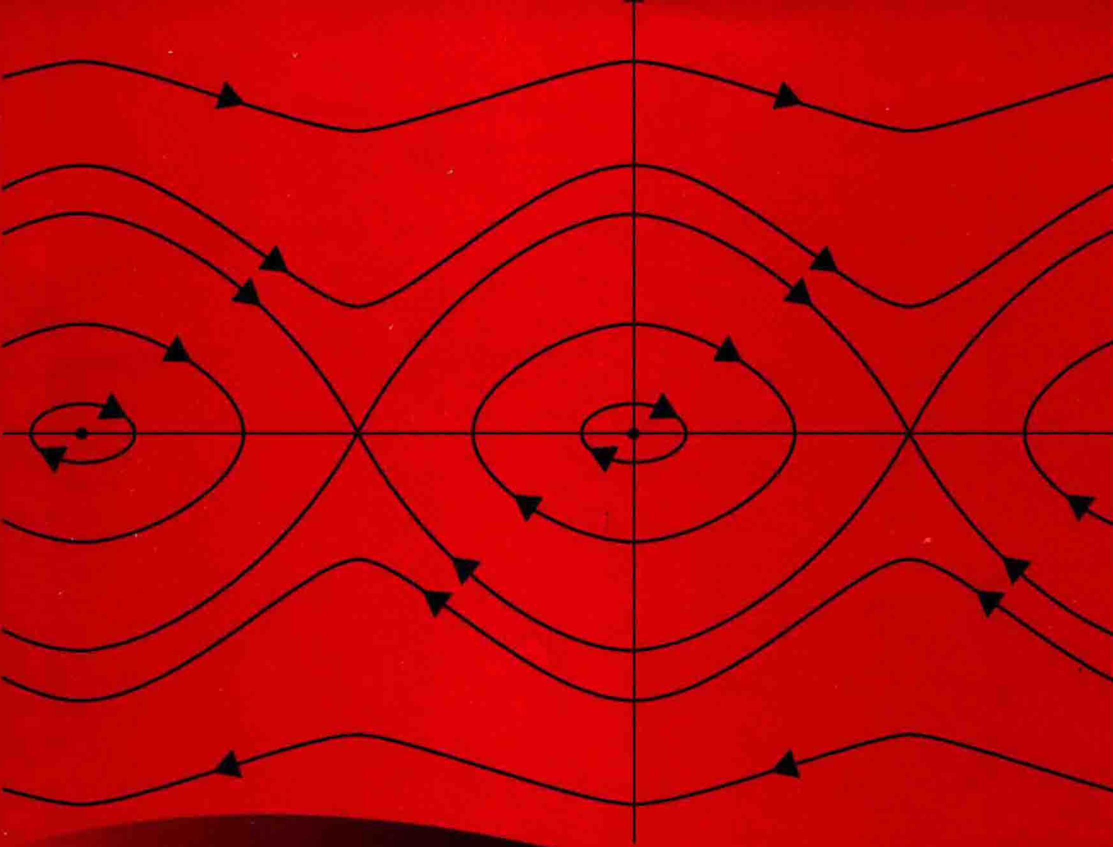

Analytical Mechanics

Antonio Fasano and
Stefano Marmi

OXFORD GRADUATE TEXTS

Analytical Mechanics

An Introduction

Antonio Fasano
University of Florence

Stefano Marmi
SNS, Pisa

Translated by
Beatrice Pelloni
University of Reading

OXFORD
UNIVERSITY PRESS

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2, 6DP,
United Kingdom

Oxford University Press is a department of the University of Oxford. It furthers the Universitys objective of excellence in research, scholarship, and education by publishing worldwide. Oxford is a registered trade mark of Oxford University Press in the UK and in certain other countries

First published in English 2006
First published in paperback 2013

Impression: 1

British Library Cataloguing in Publication Data
Data available

ISBN 978–0–19–850802–1 (hbk.)
ISBN 978–0–19–967385–8 (pbk.)

Printed and bound by
CPI Group (UK) Ltd, Croydon, CR0 4YY

Analytical Mechanics

Preface to the English Translation

The proposal of translating this book into English came from Dr. Sonke Adlung of OUP, to whom we express our gratitude. The translation was preceded by hard work to produce a new version of the Italian text incorporating some modifications we had agreed upon with Dr. Adlung (for instance the inclusion of worked out problems at the end of each chapter). The result was the second Italian edition (Bollati-Boringhieri, 2002), which was the original source for the translation. However, thanks to the kind collaboration of the translator, Dr. Beatrice Pelloni, in the course of the translation we introduced some further improvements with the aim of better fulfilling the original aim of this book: to explain analytical mechanics (which includes some very complex topics) with mathematical rigour using nothing more than the notions of plain calculus. For this reason the book should be readable by undergraduate students, although it contains some rather advanced material which makes it suitable also for courses of higher level mathematics and physics.

Despite the size of the book, or rather because of it, conciseness has been a constant concern of the authors. The book is large because it deals not only with the basic notions of analytical mechanics, but also with some of its main applications: astronomy, statistical mechanics, continuum mechanics and (very briefly) field theory.

The book has been conceived in such a way that it can be used at different levels: for instance the two chapters on statistical mechanics can be read, skipping the chapter on ergodic theory, etc. The book has been used in various Italian universities for more than ten years and we have been very pleased by the reactions of colleagues and students. Therefore we are confident that the translation can prove to be useful.

Antonio Fasano

Stefano Marmi

Contents

1 GEOMETRIC AND KINEMATIC FOUNDATIONS OF LAGRANGIAN MECHANICS

Geometry is the art of deriving good reasoning from badly drawn pictures[1]

The first step in the construction of a mathematical model for studying the motion of a system consisting of a certain number of points is necessarily the investigation of its geometrical properties. Such properties depend on the possible presence of limitations (constraints) imposed on the position of each single point with respect to a given reference frame. For a one-point system, it is intuitively clear what it means for the system to be constrained to lie on a curve or on a surface, and how this constraint limits the possible motions of the point. The geometric and hence the kinematic description of the system becomes much more complicated when the system contains two or more points, mutually constrained; an example is the case when the distance between each pair of points in the system is fixed. The correct set-up of the framework for studying this problem requires that one first considers some fundamental geometrical properties; the study of these properties is the subject of this chapter.

1.1 Curves in the plane

Curves in the plane can be thought of as *level sets* of functions $F : U \to \mathbf{R}$ (for our purposes, it is sufficient for F to be of class $\mathcal{C}^2$), where U is an open connected subset of $\mathbf{R}^2$. The curve C is defined as the set

$$C = \{(x_1, x_2) \in U | F(x_1, x_2) = 0\}. \tag{1.1}$$

We assume that this set is non-empty.

Definition 1.1 *A point P on the curve (hence such that $F(x_1, x_2) = 0$) is called* non-singular *if the gradient of F computed at P is non-zero:*

$$\nabla F(x_1, x_2) \neq 0. \tag{1.2}$$

A curve C whose points are all non-singular is called a regular curve. ∎

By the implicit function theorem, if P is non-singular, in a neighbourhood of P the curve is representable as the graph of a function $x_2 = f(x_1)$, if $(\partial F/\partial x_2)_P \neq 0$,

[1] Anonymous quotation, in Felix Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert,* Springer-Verlag, Berlin 1926.

or of a function $x_1 = f(x_2)$, if $(\partial F/\partial x_1)_P \neq 0$. The function f is differentiable in the same neighbourhood. If x_2 is the dependent variable, for x_1 in a suitable open interval I,

$$C = \text{graph}\,(f) = \{(x_1, x_2) \in \mathbf{R}^2 | x_1 \in I, x_2 = f(x_1)\}, \tag{1.3}$$

and

$$f'(x_1) = -\frac{\partial F/\partial x_1}{\partial F/\partial x_2}.$$

Equation (1.3) implies that, at least locally, the points of the curve are in one-to-one correspondence with the values of one of the Cartesian coordinates.

More generally, it is possible to use a *parametric representation* (of class $\mathcal{C}^2$) $\mathbf{x} : (a, b) \to \mathbf{R}^2$, where (a, b) is an open interval in $\mathbf{R}$:

$$C = \mathbf{x}((a, b)) = \{(x_1, x_2) \in \mathbf{R}^2 | \text{ there exists } t \in (a, b),\ (x_1, x_2) = \mathbf{x}(t)\}. \tag{1.4}$$

Note that the graph (1.3) can be interpreted as the parametrisation $\mathbf{x}(t) = (t, f(t))$, and that it is possible to go from (1.3) to (1.4) introducing a function $x_1 = x_1(t)$ of class $\mathcal{C}^2$ and such that $\dot{x}_1(t) \neq 0$.

It follows that Definition 1.1 is equivalent to the following.

DEFINITION 1.2 *If the curve C is given in the parametric form $\mathbf{x} = \mathbf{x}(t)$, a point $\mathbf{x}(t_0)$ is called* non-singular *if* $\dot{\mathbf{x}}(t_0) \neq 0$. ■

The *tangent line* at a non-singular point $\mathbf{x}_0 = \mathbf{x}(t_0)$ can be defined as the first-order term in the series expansion of the difference $\mathbf{x}(t) - \mathbf{x}_0 \sim (t - t_0)\dot{\mathbf{x}}(t_0)$, i.e. as the best linear approximation to the curve in the neighbourhood of $\mathbf{x}_0$.

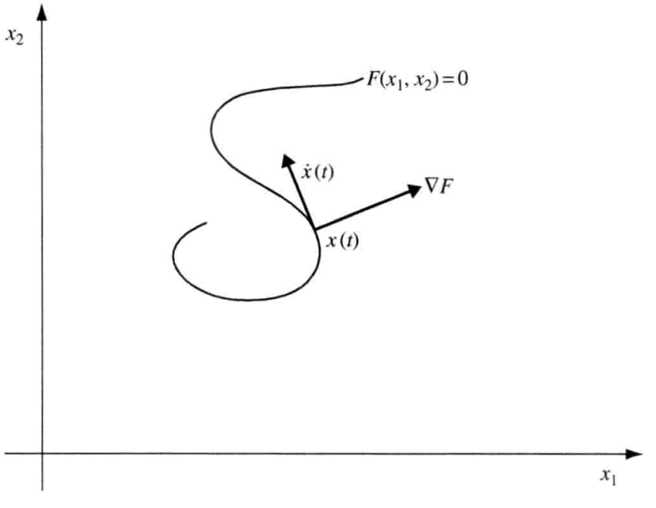

Fig. 1.1

Since $\dot{\mathbf{x}} \cdot \nabla F(\mathbf{x}(t)) = 0$, the vector $\dot{\mathbf{x}}(t_0)$, which characterises the tangent line and can be called the *velocity* on the curve, is orthogonal to $\nabla F(\mathbf{x}_0)$ (Fig. 1.1).

Example 1.1
A circle $x_1^2 + x_2^2 - R^2 = 0$ centred at the origin and of radius R is a regular curve, and can be represented parametrically as $x_1 = R \cos t$, $x_2 = R \sin t$; alternatively, if one restricts to the half-plane $x_2 > 0$, it can be represented as the graph $x_2 = \sqrt{1 - x_1^2}$. The circle of radius 1 is usually denoted $\mathbf{S}^1$ or $\mathbf{T}^1$. ∎

Example 1.2
Conic sections are the level sets of the second-order polynomials $F(x_1, x_2)$. The ellipse (with reference to the principal axes) is defined by

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1 = 0,$$

where $a > b > 0$ denote the lengths of the semi-axes. One easily verifies that such a level set is a regular curve and that a parametric representation is given by $x_1 = a \sin t$, $x_2 = b \cos t$. Similarly, the hyperbola is given by

$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} - 1 = 0$$

and admits the parametric representation $x_1 = a \cosh t$, $x_2 = b \sinh t$. The parabola $x_2 - ax_1^2 - bx_1 - c = 0$ is already given in the form of a graph. ∎

Remark 1.1
In an analogous way one can define the curves in $\mathbf{R}^n$ (cf. Giusti 1989) as maps $\mathbf{x} : (a, b) \to \mathbf{R}^n$ of class $\mathcal{C}^2$, where (a, b) is an open interval in $\mathbf{R}$. The vector $\dot{\mathbf{x}}(t) = (\dot{x}_1(t), \ldots, \dot{x}_n(t))$ can be interpreted as the velocity of a point moving in space according to $\mathbf{x} = \mathbf{x}(t)$ (i.e. along the parametrised curve).

The concept of curve can be generalised in various ways; as an example, when considering the kinematics of rigid bodies, we shall introduce 'curves' defined in the space of matrices, see Examples 1.27 and 1.28 in this chapter. ∎

1.2 Length of a curve and natural parametrisation

Let C be a regular curve, described by the parametric representation $\mathbf{x} = \mathbf{x}(t)$.

DEFINITION 1.3 *The* length l *of the curve* $\mathbf{x} = \mathbf{x}(t)$, $t \in (a, b)$, *is given by the integral*

$$l = \int_a^b \sqrt{\dot{\mathbf{x}}(t) \cdot \dot{\mathbf{x}}(t)}\, \mathrm{d}t = \int_a^b |\dot{\mathbf{x}}(t)|\, \mathrm{d}t. \tag{1.5}$$

∎

In the particular case of a graph $x_2 = f(x_1)$, equation (1.5) becomes

$$l = \int_a^b \sqrt{1 + (f'(t))^2}\,\mathrm{d}t. \tag{1.6}$$

Example 1.3
Consider a circle of radius r. Since $|\dot{\mathbf{x}}(t)| = |(-r \sin t, r \cos t)| = r$, we have $l = \int_0^{2\pi} r\,\mathrm{d}t = 2\pi r$. ■

Example 1.4
The length of an ellipse with semi-axes $a \geq b$ is given by

$$\begin{aligned} l &= \int_0^{2\pi} \sqrt{a^2 \cos^2 t + b^2 \sin^2 t}\,\mathrm{d}t = 4a \int_0^{\pi/2} \sqrt{1 - \frac{a^2 - b^2}{a^2} \sin^2 t}\,\mathrm{d}t \\ &= 4a\mathbf{E}\left(\sqrt{\frac{a^2 - b^2}{a^2}}\right) = 4a\mathbf{E}(e), \end{aligned}$$

where $\mathbf{E}$ is the complete elliptic integral of the second kind (cf. Appendix 2) and e is the ellipse eccentricity. ■

Remark 1.2
The length of a curve does not depend on the particular choice of parametrisation. Indeed, let τ be a new parameter; $t = t(\tau)$ is a $\mathcal{C}^2$ function such that $\mathrm{d}t/\mathrm{d}\tau \neq 0$, and hence invertible. The curve $\mathbf{x}(t)$ can thus be represented by

$$\mathbf{x}(t(\tau)) = \mathbf{y}(\tau),$$

with $t \in (a, b)$, $\tau \in (a', b')$, and $t(a') = a$, $t(b') = b$ (if $t'(\tau) > 0$; the opposite case is completely analogous). It follows that

$$l = \int_a^b |\dot{\mathbf{x}}(t)|\,\mathrm{d}t = \int_{a'}^{b'} \left|\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}(t(\tau))\right| \left|\frac{\mathrm{d}t}{\mathrm{d}\tau}\right| \mathrm{d}\tau = \int_{a'}^{b'} \left|\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\tau}(\tau)\right| \mathrm{d}\tau.$$ ■

Any differentiable, non-singular curve admits a *natural parametrisation* with respect to a parameter s (called the *arc length, or natural parameter*). Indeed, it is sufficient to endow the curve with a positive orientation, to fix an origin O on it, and to use for every point P on the curve the length s of the arc OP (measured with the appropriate sign and with respect to a fixed unit measure) as a coordinate of the point on the curve:

$$s(t) = \pm \int_0^t |\dot{\mathbf{x}}(\tau)|\,\mathrm{d}\tau \tag{1.7}$$

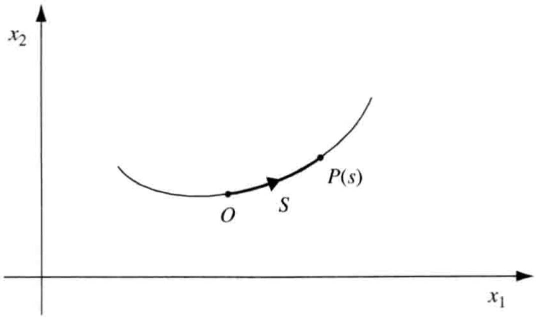

Fig. 1.2

(the choice of sign depends on the orientation given to the curve, see Fig. 1.2). Note that $|\dot{s}(t)| = |\dot{\mathbf{x}}(t)| \neq 0$.

Considering the natural parametrisation, we deduce from the previous remark the identity

$$s = \int_0^s \left| \frac{d\mathbf{x}}{d\sigma} \right| d\sigma,$$

which yields

$$\left| \frac{d\mathbf{x}}{ds}(s) \right| = 1 \quad \text{for all } s. \tag{1.8}$$

Example 1.5
For an ellipse of semi-axes $a \geq b$, the natural parameter is given by

$$s(t) = \int_0^t \sqrt{a^2 \cos^2 \tau + b^2 \sin^2 \tau}\, d\tau = 4a\mathbf{E}\left(t, \sqrt{\frac{a^2 - b^2}{a^2}}\right)$$

(cf. Appendix 2 for the definition of $\mathbf{E}(t, e)$). ■

Remark 1.3
If the curve is of class $\mathcal{C}^1$, but the velocity $\dot{\mathbf{x}}$ is zero somewhere, it is possible that there exist singular points, i.e. points in whose neighbourhoods the curve cannot be expressed as the graph of a function $x_2 = f(x_1)$ (or $x_1 = g(x_2)$) of class $\mathcal{C}^1$, or else for which the tangent direction is not uniquely defined. ■

Example 1.6
Let $\mathbf{x}(t) = (x_1(t), x_2(t))$ be the curve

$$x_1(t) = \begin{cases} -t^4, & \text{if } t \leq 0, \\ t^4, & \text{if } t > 0, \end{cases}$$

$$x_2(t) = t^2,$$