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22

Early Fourier Analysis

Hugh L. Montgomery

American Mathematical Society

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Providence, Rhode Island

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Dedicated with love to my children —

Lowell and Peter,

Hal and Eve,

Philip and Katharina

Preface

While such subjects as number theory and probability theory are commonly offered to undergraduates, it seems that Fourier analysis is rarely found, which is shocking when one considers the value of this subject not just within mathematics but also in the physical sciences and engineering. The author hopes that this book will encourage the view that Fourier analysis can be fruitfully presented not just to undergraduates, but even to younger undergraduates with no more experience than three or four terms of calculus. Such students will find a gentle introduction to the art of writing proofs and will be better prepared for advanced calculus and complex variables.

A student who has taken a course in advanced calculus may wonder what can be done with that machinery. The answer is: harmonic analysis (among other things). Paul Halmos is reported to have said words to the effect that the tragedy of Fourier series is that they were invented (in 1807) before convergence. The wonderful thing is that analysts such as Cauchy, Dirichlet, Riemann, and Weierstrass were motivated to develop the foundations of real analysis in order to make sense of Fourier series. In particular, Riemann defined his integral in order to provide a more rigorous basis for the discussion of Fourier series.

This book could be used for a capstone course of an undergraduate program or for beginning graduate students as a way to motivate the study of the Lebesgue integral. Since it is hoped that this book will be useful at a wide range of levels, it contains far more material than would ever be used in a single one term course. The author will be happy to provide suggestions adjusted to the instuctor's purpose.

We study Fourier analysis in three important settings. First we consider the Discrete Fourier Transform, which has to do with the use of roots of unity to describe periodic sequences. The results in this setting are easily obtained, and they form a framework for our endeavors in the more difficult subsequent settings. The point is that in the discrete setting there is no issue of convergence, but with Fourier Series we discover that convergence is a delicate matter. With Fourier Transforms of functions defined on the real line, matters are similar, but with additional difficulties. In the two latter situations we encounter points in our arguments where a detail is needed from advanced calculus or Lebegue measure theory. On such occasions, we simply quote the needed result and move on.

On the subject of Fourier Series, some authors use $\cos nx$ and $\sin nx$, so that all functions have period 2π. The consequence of this prescription is that most formulas have a $1/(2\pi)$ or $1/\pi$. Our contention is that the subject is more elegant when one works with functions with period 1, so that the basic building blocks are $\cos 2\pi nx$ and $\sin 2\pi nx$. But $\cos 2\pi nx + i \sin 2\pi nx = e^{2\pi inx}$ (a fact that will be a subject of discussion in Chapter 1), and it is more elegant still to use the complex exponential rather than sines and cosines. Of course, to proceed in this way, one must first become more comfortable with complex numbers. Hence that is the topic of Chapter 1. In general, when we are faced with a function with some strange period, we make a linear change of variable so that everything is translated into issues of functions with period 1. If sines and cosines are involved, we may convert to complex exponentials. When we resolve whatever is at issue, we may convert back to sines and cosines, if we wish. This is a little reminiscent of a problem expressed in terms of pounds and feet, which we would convert to grams and meters, and then convert back after the calculation is done.

Fourier analysis has links to many other branches of mathematics. We occasionally make remarks relating to such topics as linear algebra, probability theory, or number theory. Such digressions may be safely ignored by readers who are unfamiliar with the related subject in question.

Among the following chapters, sections, and appendices are found several valuable topics that are rarely found in the undergraduate (and sometimes even the graduate) curriculum. These include linear recurrences (in §F.4), summability theory (in §4.3), Bernoulli polynomials and Euler–Maclaurin summation (in §9.5), uniform distribution (in §9.6), Chebyshev polynomials (in Appendix C), and inequalities (in Appendix I).

The author is indebted to colleagues Al Taylor and Jack Goldberg for initiating this educational experiment and to the late Curtis Huntington for his unwavering support. In addition, the author is happy to thank Dick Askey, John Benedetto, Edward Crane, Peter Duren, Emily Holt, Alex Iosevich, Michael Kelly, Harsh Mehta, Kristen Moore, Michael Mossinghoff, Chris Nixon, Olivier Ramaré, Elmer Rees, Babar Saffari, and Jeff Vaaler for their valuable contributions. It has been a pleasure to work with editor Sergei Gelfand and his competent and attentive support staff at the AMS. Finally, the author thanks Michele MacFarlane, who cheerfully accepted a double dose of domestic chores in order that the author would have more time to write.

Hugh L. Montgomery
Ann Arbor, Michigan
August 31, 2014

Contents

Background

In the first section below, terms are defined and proofs are included. In the second section, most terms are defined, but proofs are omitted. In the third section, not only are proofs omitted, but some terms are undefined. The point is that everything we do can be made fully rigorous, but neither lack of rigor nor absence of advanced training in analysis should interfere with the acquisition of Fourier analysis in its most classical settings.

0.1. Elementary mathematics

An *arithmetic progression* (sometimes abbreviated AP) is a set of the form $\{nq+a : \in \mathbb{Z}\}$. Hence a sequence $\{u_n\}$ is said to be in arithmetic progression if $u_{n+1} - u_n$ is the same for all n. That means that $u_n = nq + a$ for some q and a. We frequently sum such numbers.

Theorem 0.1. *If $u_1, u_2, \ldots, u_N$ are consecutive members of an arithmetic progression, then*

$$u_1 + u_2 + \cdots + u_N = N \cdot \frac{u_1 + u_N}{2}. \tag{0.1}$$

For example,

$$1 + 2 + \cdots + N = \frac{N(N+1)}{2}.$$

Proof. Let d be determined so that $u_{n+1} - u_n = d$ for all n. We write the sum twice, first in its natural order and then in reverse order. Thus if S is the sum, then

$$\begin{aligned} S &= u_1 + u_2 + u_3 + \cdots + u_{N-1} + u_N, \\ S &= u_N + u_{N-1} + u_{N-2} + \cdots + u_2 + u_1. \end{aligned}$$

We now sum in columns. On the left hand side we have $2S$. In the first column on the right we have $u_1 + u_N$. In the second column on the right we have $u_2 + u_{N-1} = (u_1 + d) + (u_N - d) = u_1 + u_N$. In the third column we have $u_3 + u_{N-2} =$

$(u_2+d)+(u_{N-1}-d) = u_2+u_{N-1} = u_1+u_N$. Continuing in this way, we find that $u_k+u_{N+1-k} = u_1+u_N$ for all k. Hence the right hand side above is $N(u_1+u_N)$, so we have the result. □

A sequence $\{u_n\}$ is said to be a *geometric progression* if u_{n+1}/u_n is the same for all n. That means that $u_n = ar^n$ for some a and r.

Theorem 0.2. *The sum of a geometric progression is*

$$1+r+r^2+\cdots+r^{N-1} = \begin{cases} \dfrac{1-r^N}{1-r} & (r \neq 1), \\ N & (r=1). \end{cases} \tag{0.2}$$

If the first term to be summed is a power of r, we can simply factor out that amount. Thus for example,

$$\sum_{n=M}^{M+N-1} r^n = \frac{r^M - r^{M+N}}{1-r}$$

if $r \neq 1$. If $|r|<1$, then $r^N \to 0$ as $N \to \infty$, so

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}. \tag{0.3}$$

Proof. If $r=1$, then each of the N terms on the left is 1, so the sum is N. Suppose that $r \neq 1$, and let S denote the sum. Then

$$rS = r + r^2 + r^3 + \cdots + r^N.$$

On subtracting this from S we see that most terms cancel, leaving

$$S - rS = 1 - r^N.$$

The stated formula now follows on dividing both sides by $1-r$. □

If $p(x) = ax^2+bx+c$ is a quadratic polynomial, then

$$4ap(x) = 4a^2x^2 + 4abx + 4ac = (2ax+b)^2 + 4ac - b^2.$$

This manipulation is called *completing the square.* If $p(x)=0$, then $(2ax+b)^2 = d$ where $d = b^2-4ac$ is the *discriminant* of the polynomial. If $d>0$, then the equation $p(x)=0$ has two distinct real roots, namely

$$r_1 = \frac{-b+\sqrt{d}}{2a}, \qquad r_2 = \frac{-b-\sqrt{d}}{2a}.$$

If $d=0$, then $p(x)$ has a double root, $r_1 = r_2 = -b/(2a)$. If $d<0$, then $p(x)$ has no real root, but it has two complex roots,

$$\frac{-b \pm i\sqrt{-d}}{2a}.$$

In all three cases the sum of the roots is $-b/a$, the product of the roots is c/a, and the polynomial factors, $p(x) = a(x-r_1)(x-r_2)$.

Let $a_1, a_2, \ldots, a_N$ and $b_1, b_2, \ldots, b_N$ be real numbers. *Cauchy's Inequality* asserts that

$$\Big|\sum_{n=1}^{N} a_n b_n\Big| \le \Big(\sum_{n=1}^{N} a_n^2\Big)^{1/2}\Big(\sum_{n=1}^{N} b_n^2\Big)^{1/2}. \tag{0.4}$$

That this is so follows immediately from the algebraic identity

$$\Big(\sum_{n=1}^{N} a_n^2\Big)\Big(\sum_{n=1}^{N} b_n^2\Big) - \Big(\sum_{n=1}^{N} a_n b_n\Big)^2 = \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N}(a_m b_n - a_n b_m)^2,$$

since a sum of squares of real numbers is nonnegative. Moreover, from the above identity it is clear that equality holds in (0.4) if and only if the two sequences $\{a_n\}$, $\{b_n\}$ are proportional.

The *Principle of Mathematical Induction* is one of the axioms that define the integers. It can be formulated in a number of (equivalent) ways.

(1) *Weak induction*: If S is a set of positive integers, if $1 \in S$, and if $n+1 \in S$ whenever $n \in S$, then S is the set of all positive integers.

(2) *Strong induction*: If S is a set of positive integers, if $1 \in S$, and if $k \in S$ for all positive integers $k < n$ implies that $n \in S$, then S is the set of all positive integers.

(3) *Well ordering*: If S is a set of positive integers, and S is non-empty, then S contains a least member.

In all three cases we are inducting from 1, but of course one could instead induct from 0 or any other convenient point.

The Binomial Theorem is treated in Appendix B. A catalogue of trigonometric formulæ is provided in Appendix T, for convenience. The manner in which we express $\cos n\theta$ as a polynomial in $\cos\theta$ is the subject of Appendix C.

0.2. Real analysis

It is not our purpose to summarize all of real analysis. We mention only specific items that we need, and these are largely concerned with such issues as conditions that ensure that (a) one can exchange two limiting operations; and (b) a sequence that appears to tend to a limit does so.

Theorem 0.3. *A bounded monotonic sequence of real numbers has a limit.*

A sequence $\{x_n\}$ is said to be a *Cauchy sequence*if

$$\lim_{\substack{m\to\infty \\ n\to\infty}} (x_m - x_n) = 0. \tag{0.5}$$

Clearly any sequence that tends to a finite limit is a Cauchy sequence. What is important is that the converse is also true:

Theorem 0.4. *If $\{x_n\}$ is a Cauchy sequence, then $\lim_{n\to\infty} x_n$ exists and is finite.*

Note that a sequence of rational numbers tending to $\sqrt{2}$ is a Cauchy sequence, but does not not have a limit within the system of rational numbers, because $\sqrt{2}$ is irrational. In a set-theoretic sense, the real numbers are constructed by filling in the holes found among the rational numbers. Because all Cauchy sequences have a limit, we say that the real numbers are *complete.*

A function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$. That is, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ if $|x - a| < \delta$. Here the choice of δ depends on both ε and a. However, if in a domain $\mathcal{D}$ we have $|f(x) - f(y)| < \varepsilon$ whenever $x \in \mathcal{D}$, $y \in \mathcal{D}$ and $|x - y| < \delta$, then we say that f is *uniformly continuous* on $\mathcal{D}$.

Theorem 0.5. *If a real-valued function $f(x)$ is continuous on a closed bounded interval $[a, b]$, then it is uniformly continuous on that interval, and attains its maximum and minimum values.*

The same theorem also holds for real-valued continuous functions defined on closed bounded sets in the plane $\mathbb{R}^2$.

Theorem 0.6. (The Squeeze Theorem) *Suppose that $\delta > 0$, that f_-, f, and f_+ are functions such that $f_-(x) \le f(x) \le f_+(x)$ for $a - \delta < x < a + \delta$. If $\lim_{x \to a} f_-(x) = c$ and $\lim_{x \to a} f_+(x) = c$, then $\lim_{x \to a} f(x) = c$.*

Theorem 0.7. (Rolle's Theorem) *If $f(x)$ is a continuous real-valued function on the interval $a \le x \le b$ with $f(x)$ differentiable for $a < x < b$, and if $f(a) = f(b)$, then there exists a $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

Theorem 0.8. (The Mean Value Theorem of Differential Calculus) *If $f(x)$ is a continuous real-valued function on the interval $a \le x \le b$ and $f(x)$ is differentiable for $a < x < b$, then there exists a $\xi \in (a, b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}\,.$$

Theorem 0.9. *If $\sum_{n=1}^{\infty} |a_n| < \infty$, then the sum $\sum_{n=1}^{\infty} a_n$ converges.*

Theorem 0.10. *Suppose that $a_{mn} \ge 0$ for all m and n. We form two sums:*

$$\sum_{m=1}^{\infty} \Big(\sum_{n=1}^{\infty} a_{mn} \Big), \qquad \sum_{n=1}^{\infty} \Big(\sum_{m=1}^{\infty} a_{mn} \Big). \tag{0.6}$$

If either of these sums is finite, then the other one is also finite, and they are equal.

Theorem 0.11. *If $|a_{mn}| \le A_{mn}$ for all m and n, and if*

$$\sum_{m=1}^{\infty} \Big(\sum_{n=1}^{\infty} A_{mn} \Big) < \infty,$$

then the sums (0.6) converge and are equal.

Theorem 0.12. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series, and let R be defined by the relation*

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n}\,.$$

Then the power series is convergent for $|z| < R$, and is divergent for $|z| > R$. For $|z| < R$ a power series may be differentiated term-by-term, and the differentiated power series has the same radius of convergence R.

To define what it means to say that a function is Riemann–integrable on an interval $[a, b]$, we start with a partition $\boldsymbol{\pi}$, which is to say a sequence $\{x_n\}$ such that

$$a = x_0 < x_1 < \cdots < x_J = b,$$

and choose interspersing numbers ξ_j so that

$$a = x_0 \le \xi_1 \le x_1 \le \xi_2 \le x_2 \le \cdots \le \xi_{J-1} \le x_{J-1} \le \xi_J \le x_J = b. \tag{0.7}$$

A *Riemann sum* for $\int_a^b f(x)\,dx$ is then a sum of the form

$$S(\boldsymbol{\pi}, \boldsymbol{\xi}) = \sum_{j=1}^{J} f(\xi_j)(x_j - x_{j-1})\,.$$

The *mesh* of $\boldsymbol{\pi}$ is defined to be

$$\mathrm{mesh}(\boldsymbol{\pi}) = \max_{1 \le j \le J} (x_j - x_{j-1})\,. \tag{0.8}$$

That is, the mesh is the length of the longest subinterval defined by $\boldsymbol{\pi}$. We say that the integral exists and has the value I if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\mathrm{mesh}(\boldsymbol{\pi}) < \delta$, then $|S(\boldsymbol{\pi}, \boldsymbol{\xi}) - I| < \varepsilon$ for any choice of the interspersing points $\boldsymbol{\xi}$.

As to sums and integrals such as

$$\sum_{a \le n \le b} u_n \quad \text{and} \quad \int_a^b f(x)\,dx,$$

we have two different conventions. In a sum, we sum over all n that satisfy the indicated constraints. Thus if $b < a$, then there is no such n, and the value of the sum is 0. However, for integrals, if $b < a$, we simply say that the value of the integral is $-\int_b^a f(x)\,dx$.

The arc length of a parameterized curve $(x(t), y(t))$ for $a \le t \le b$ is the supremum of all sums of the form

$$\sum_{j=1}^{J} \sqrt{(x(t_j) - x(t_{j-1}))^2 + (y(t_j) - y(t_{j-1}))^2} \tag{0.9}$$

where $a = t_0 < t_1 < \cdots < t_J = b$.

Theorem 0.13. (Fundamental Theorem of Calculus, First Form) *Suppose that $f(x)$ is Riemann-integrable on the interval $[a, b]$. For $a \le x \le b$, put*

$$F(x) = \int_a^x f(u)\,du\,.$$

Then $F(x)$ is continuous on the interval $[a, b]$. If $a < c < b$ and if $f(x)$ is continuous at $x = c$, then $F(x)$ is differentiable at $x = c$, and $F'(c) = f(c)$.

Theorem 0.14. (Fundamental Theorem of Calculus, Second Form) *Suppose that $f(x)$ is Riemann-integrable on the interval $[a,b]$. Suppose further that $F(x)$ is a differentiable function on the interval $[a,b]$ such that $F'(x) = f(x)$ for $a \le x \le b$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The Second Form of the Fundamental Theorem follows from the First Form if f is continuous. The point of the Second Form is that it holds under the weaker assumption that f is Riemann-integrable.

Theorem 0.15. (Integration by parts) *Suppose that f is Riemann-integrable on $[a,b]$, that F is a function such that $F'(x) = f(x)$, and that g is a differentiable function such that $g'(x)$ is Riemann-integrable on $[a,b]$. Then*

$$\int_a^b f(x)g(x)\,dx = F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)\,dx.$$

This follows immediately from the Second Form of the Fundamental Theorem, in view of the differentiation formula $(Fg)' = F'g + Fg' = fg + Fg'$.

Theorem 0.16. (The triangle inequality for integrals) *If f is Riemann-integrable, then*

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Theorem 0.17. (Leibniz's Rule) *If $f(x,y)$ and $\frac{\partial}{\partial x}f(x,y)$ exist and are continuous on the closed rectangle $a \le x \le b$, $c \le y \le d$, then the function*

$$F(x) = \int_c^d f(x,y)\,dy$$

is differentiable for $a \le x \le b$, and

$$F'(x) = \int_c^d \frac{\partial}{\partial x} f(x,y)\,dy.$$

Theorem 0.18. *Suppose that $f(x) = \sum_{n=1}^{\infty} f_n(x)$. If the functions f_n are differentiable, and if the series $\sum_{n=1}^{\infty} f_n'(x)$ is uniformly convergent, then f is differentiable, and*

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x).$$

Theorem 0.19. (Dominated Convergence) *If a_{mn} is a double sequence, and A_n is such that $\lim_{m\to\infty} a_{mn} = A_n$ exists, and if there is a sequence M_n such that $|a_{mn}| \le M_n$ for all m, and $\sum_{n=1}^{\infty} M_n < \infty$, then*

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} = \sum_{n=1}^{\infty} A_n.$$