Series in Information and Computational Science — 41

Adaptive Finite Element Methods for Optimal Control Governed by PDEs

Wenbin Liu Ningning Yan

（偏微分方程最优控制的自适应有限元方法）

SCIENCE PRESS
Beijing

(O—3230.0101)

Series in Information and Computational Science

ISBN 978-7-03-022256-5

定 价：46.00 元

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Responsible Editor: Fan Qingkui

Copyright© 2008 by Science Press
Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, China

Printed in Beijing

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ISBN 978-7-03-022256-5 (Beijing)

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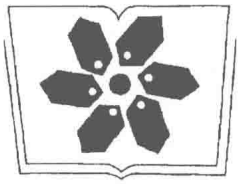

中国科学院科学出版基金资助出版

Series in Information and Computational Science 41

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Preface to the Series in Information and Computational Science

Since the 1970s, Science Press has published more than thirty volumes in its series Monographs in Computational Methods. This series was established and led by the late academician, Feng Kang, the founding director of the Computing Center of the Chinese Academy of Sciences. The monograph series has provided timely information of the frontier directions and latest research results in computational mathematics. It has had great impact on young scientists and the entire research community, and has played a very important role in the development of computational mathematics in China.

To cope with these new scientific developments, the Ministry of Education of the People's Republic of China in 1998 combined several subjects, such as computational mathematics, numerical algorithms, information science, and operations research and optimal control, into a new discipline called Information and Computational Science. As a result, Science Press also reorganized the editorial board of the monograph series and changed its name to Series in Information and Computational Science. The first editorial board meeting was held in Beijing in September 2004, and it discussed the new objectives, and the directions and contents of the new monograph series.

The aim of the new series is to present the state of the art in Information and Computational Science to senior undergraduate and graduate students, as well as to scientists working in these fields. Hence, the series will provide concrete and systematic expositions of the advances in information and computational science, encompassing also related interdisciplinary developments.

I would like to thank the previous editorial board members and assistants, and all the mathematicians who have contributed significantly to the monograph series on Computational Methods. As a result of their contributions the monograph series achieved an outstanding reputation in the community. I sincerely wish that we will extend this support to the new Series in Information and Computational Science, so that the new series can equally enhance the scientific development in information and computational science in this century.

Shi Zhongci

2005.7

Preface

Finite element method has been widely used in computing numerical solutions of all kinds of distributed optimal control problems. Although there were several excellent monographs on finite element approximation of distributed optimal control, to our best knowledge this is among the first books to cover adaptive finite element method for optimal control. It has been recently found that suitable adaptive meshes can greatly reduce discretization errors of finite element approximation of optimal control. If the computational meshes are not properly generated, then there may be large error around the singularities of the control, which cannot be removed later on. It was noticed recently that the error indicators derived for the approximation of the state equation, which have been widely used in adaptive finite element schemes for optimal control, are not necessarily efficient for computing the OPTIMAL CONTROL, and purposed built error estimators for the control problem are essential in such schemes. The purpose of this monograph is to provide an introduction to adaptive finite element approximation of distributed optimal control problems, which has recently received increasing attention. The book should be specially suitable for students who undertake research related to computational issues of optimal control.

In this book we emphasize discussions of unique issues from the adaptive finite element approximation of optimal control. The main idea used in error analysis is to first combine convex analysis and interpolation error estimations of suitable interpolators, which much depend on the structure of the control constraints, in order to derive error estimates for the control via the variational inequalities in the optimality conditions, and then to apply the standard techniques for deriving error estimates for the state equations. The need, the framework and the techniques of using multi adaptive meshes in developing efficient numerical algorithms for optimal control have been emphasized throughout the book. We started from several typical examples of optimal control problems and then discussed existence and optimality conditions for some optimal control problems. We believe that these discussions are especially useful for the researchers and students who first enter this area. Then we set up finite element approximation schemes for several typical optimal control problems and proceeded to discuss their a priori and a posteriori error estimates, and to study numerical algorithms.

In Chapter 1 we present some model optimal control problems. Optimal control problems in real applications often have complex formulations and need much extra knowledge to understand. In Chapter 1 we try to present some typical control problems with simplified structure so that the readers should be able to understand the main features of

these optimal control problems more easily. In Section 1 we introduce optimal control problems governed by elliptic equations, including parameter identification and coefficient control problems, and then those governed by evolutional equations in Section 2. Afterwards we introduce some simple flow control problems and optimal shape design, although the latter has not been studied further in this book.

In Chapter 2 existence and optimality conditions have been discussed. Existence of optimal control is straightforward to establish for convex optimal control problems. However it can be very involved to discuss for optimal shape design and coefficient control, or for these problems with complex state equations like the full N-S equations. Furthermore existence is not the primer concern of our investigation. Thus we only illustrate the basic ideas and techniques through a proof of existence of a parameter identification. On the other hand, optimality conditions are very important in developing numerical schemes for optimal control, which are derived via differentiating objective functional. Thus we start with introduction of directional derivative of functional and examples of several functional. Then we state an abstract optimality condition and show how to derive optimality conditions of linear elliptic optimal control. Then we further establish optimality conditions for more complicated control problems including parameter identification.

In Chapter 3, we set up some basic finite element approximation schemes for optimal control problems. We firstly give the standard finite element approximations and then the mixed finite element approximation schemes for elliptic optimal control problems. We then introduce finite element schemes for flow control governed by the Stokes equations. Finally we investigate the finite element approximation of boundary control, where the control only acts on the boundary of the domain.

In Chapters 4-5, we investigate a priori error estimates for these finite element schemes introduced in Chapter 3 using a uniform approach. We first establish a prior error estimates of optimal order for the standard finite element approximation schemes of the elliptic optimal control and then the boundary control. Then we study superconvergence properties that are important in finite element approximation theory. Due to lower order regularity of optimal control (normally only in H^1), different approaches have to be adopted in dealing with superconvergence of optimal control. Then we study these topics for the mixed finite element approximation of the elliptic control and the Stokes flow control in the rest of the chapters.

In Chapter 6, some basic ideas of adaptive finite element method are first introduced. As a preparation for adaptive finite element method of optimal control, both lower and upper a posteriori error estimates of residual type are derived for the finite element approximation of the linear elliptic equations. A posteriori error estimators of recovery type are further studied again for the elliptic equations. Then some unique issues on adaptive finite element approximation for optimal control are addressed. Some current approaches are discussed. Motivations and necessities of using purposed built error indicators for the approximation of the optimal control, and multi adaptive meshes are discussed.

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In Chapter 7, we follow the basic ideas and discussions in Chapter 6 and start to develop purposed built a posteriori error estimates for the standard finite element approximation of constrained elliptic optimal control problems. We present very detailed descriptions for our approach, which combines convex analysis and the standard techniques of deriving a posteriori error estimates of residual type. The main technical difficult is how to handle the variational inequalities in the coupled optimality conditions. We derive both lower and upper a posteriori error estimates in different norms. We further derive sharp a posteriori error estimates for the boundary control. These estimates are of local character in the sense they are mainly affected by the states approximation error near the boundary.

In Chapter 8, an overview of numerical methods for constrained optimal control is first presented. Then a preconditioned gradient method of projection type is introduced and discussed. Explicit numerical solutions of variational inequalities in the optimal conditions are derived. This algorithm is further applied to several examples of optimal control and results of numerical tests are presented. Among others, these results show the multi-meshes are indeed important in developing efficient finite element algorithms for constrained optimal control.

In Chapter 9, a posteriori error estimators of recovery type are investigated. The recovery operators for the optimal control are different from those used in standard finite element approximation of elliptic equations as here we only recover the function value of optimal control although gradient also needs recovering for the states. Recovery operators and then a posteriori error estimators are constructed and investigated. Lower and upper a posteriori error bounds are established. Finally these error estimators are shown to be asymptotically exact under certain regularity conditions.

In Chapter 10 we discuss a posteriori error estimates for the mixed finite element approximation of optimal control. We first discuss the mixed finite element method for the elliptic optimal control problems. There are many cases where gradient is included in the objective functional. In such a case, the mixed finite element method is quite suitable as it gives higher accuracy for the gradient approximation although it is much more involved to analysis. We then investigate the mixed finite element schemes for Stokes flows, and derive lower and upper a posteriori error estimates. The techniques used can be extended to full Navier-Stokes equations as long as the Reno number Re is not too big.

We wish to express our sincere thanks to Prof. Shi Z.C., who kindly agrees to include this book into his series. Our thanks go to Prof. Tang Tao, who encouraged us to write this book at the first place. We also wish to thank our collaborators and friends, Prof. Chen Y.P., Porf. Chen W., Prof. Hoppe, Prof. Huang Y.Q., Prof. Li R., Prof. Lu T.C., Prof. Ma H.P., Prof. Rui H.X., Prof. Yang D.P., Dr. Liu H.P. and Gong W. Without their help and collaboration, it is impossible for us to complete this book.

We welcome comments and corrections to the book and can be reached at

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Writing this book has been supported by the National Basic Research Program under the Grant 2005CB321701 and the National Natural Science Foundation of China (No. 60474027 and 10771211). It is also supported in part by the Outstanding Overseas Chinese Scholars Fund of Chinese Academy of Sciences. A part of this book was finished while the first author was visiting Department of Mathematics, Hong Kong Baptist University, and School of Mathematics, Shandong University. He wishes to express his sincere thanks for the stimulating discussions with the staff and the students and the generous supports from the schools.

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Chapter 1

Introduction

Optimal control problems have been so widely met in all kinds of practical problems. It is impossible to present any comprehensive review on their applications here. In this chapter we mention some model examples of optimal control problems. Most of the optimal control problems that we are interested in can be symbolically written in the following form: (OCP)

$$\min_{u \in U_{ad}} J(y, u)$$

subject to

$$A(y, u) = 0,$$

where

J is the objective functional;

y is the state variable in a state space V;

u is the control variable in a control space U;

U_{ad} is the admissible set of the control (design);

A is a suitable operator on $V \times U_{ad}$.

In general the state equation $A(y, u) = 0$ can be a PDE, or a variational inequality, etc., and may incorporate state constraints. More details will be given below.

Thus the mechanism is to control the state (like temperature or position of a system, usually not easy to control directly) through the control variable, which drives the state through the state equation. Sometimes, the control variable is called design variable, the control space is referred to as design space, and the state variable may be referred to as system variable.

Let Ω (Ω_U) be a bounded open set in $\mathcal{R}^n$ ($n \leqslant 3$) with the Lipschitz boundary $\partial\Omega$ ($\partial\Omega_U$). In this book we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{W^{m,q}(\Omega)}$ (or $\|\cdot\|_{m,q,\Omega}$) and seminorm $|\cdot|_{W^{m,q}(\Omega)}$ (or $|\cdot|_{m,q,\Omega}$). We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : \partial^\alpha w|_{\partial\Omega} = 0,\ |\alpha| \leqslant m-1\}$. We denote $W^{m,2}(\Omega)$ ($W_0^{m,2}(\Omega)$) by $H^m(\Omega)$ ($H_0^m(\Omega)$) with norm $\|\cdot\|_{H^m(\Omega)}$ (or $\|\cdot\|_{m,\Omega}$). In addition c or C denotes a general positive constant.

1.1 Examples of optimal control for elliptic systems

Let the state space $V = H_0^1(\Omega)$, the control space $U = L^2(\Omega)$. Let $y \in V$ be the temperature distribution in the body Ω, which satisfies the elliptic equation:

$$-\mathrm{div}(A\nabla y) = Bu \quad \text{in } \Omega, \quad y|_{\partial\Omega} = 0,$$

where A is conductivity matrix and Bu represents heat resource density inside the body, which can be controlled by $u \in U_{ad}$, where U_{ad} is a closed convex set in U. For example, often $(Bu)(x) = c(x)u(x)$, where $c(x)$ is the density factor. Assume that the body's surface temperature is fixed, say, zero. The aim of the control is to make the temperature distribution y as close as to the desirable distribution $z_d \in V$, e.g., $z_d = 100$. Of course there could be many controls to achieve this objective, but we wish to achieve this using as small as possible energy, which can be represented by the formula $\int_\Omega u^2$. Then we have the following Static Temperature Control Problem:

$$\min_{u\in U_{ad}} \left\{ \frac{1}{2} \left(\int_\Omega (y - z_d)^2 dx + \int_\Omega u^2 dx \right) \right\} \tag{1.1.1}$$

subject to

$$-\mathrm{div}(A\nabla y) = f + Bu \quad \text{in } \Omega, \quad y|_{\partial\Omega} = 0.$$

Let us recall that in the optimal control problem, y and u are called the state and control respectively, the equation is called state equation, the functional to be optimized is called objective functional, and U_{ad} is called the control constraint set.

Assume that the system can heat or cool the body. Then there is no constraints for u so $U_{ad} = U$, and the control problem is called un-constrained control problem. If we can only heat the body, then there will be control constraint like: $u \geqslant 0$ with the admissible control set: $U_{ad} = \{u : u \geqslant 0\}$. Then the control problem is called constrained control problem.

For more general control purposes, the objective functional can be other than quadratic. In this book we will assume the state and control can be separated. Then we have the general optimal control problem:

$$\min_{u\in U_{ad}} \{g(y) + j(u)\} \tag{1.1.2}$$
$$-\mathrm{div}(A\nabla y) = f + Bu \ \text{ in } \Omega, \quad y|_{\partial\Omega} = 0,$$

where B is a linear operator, g and j are strictly convex functionals and U_{ad} is a closed convex set in the control space U.

For example, $U_{ad} = \left\{ u : \int_{\Omega_U} u^2 \leqslant 1 \right\}$, or $U_{ad} = \{u : -1 \leqslant u \leqslant 1\}$. Sometimes, there are further constraints for the state like $y \geqslant 0$ – the so called control problem with state constraints. This class of control problems is often complicated so will be omitted in this book.

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