Graduate Texts in Mathematics

J.L. Doob

Measure Theory

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Introduction

This book was planned originally not as a work to be published, but as an excuse to buy a computer, incidentally to give me a chance to organize my own ideas on what measure theory every would-be analyst should learn, and to detail my approach to the subject. When it turned out that Springer-Verlag thought that the point of view in the book had general interest and offered to publish it, I was forced to try to write more clearly and search for errors. The search was productive.

Readers will observe the stress on the following points.

The application of pseudometric spaces. Pseudometric, rather than metric spaces, are applied to obviate the artificial replacement of functions by equivalence classes, a replacement that makes the use of "almost everywhere" either improper or artificial. The words "function" and "the set on which a function has values at least ε" can be taken literally in this book. Pseudometric space properties are applied in many contexts. For example, outer measures are used to pseudometrize classes of sets and the extension of a finite measure from an algebra to a σ algebra is thereby reduced to finding the closure of a subset of a pseudometric space.

Probability concepts are introduced in their appropriate place, not consigned to a ghetto. Mathematical probability is an important part of measure theory, and every student of measure theory should be acquainted with the fundamental concepts and function relations specific to this part. Moreover, probability offers a wide range of measure theoretic examples and applications both in and outside pure mathematics. At an elementary level, probability-inspired examples free students from the delusions that product measures are the only important multidimensional measures and that continuous distributions are the only important distributions. At a more sophisticated level, it is absurd that analysts should be familiar with mutual orthogonality but not with mutual independence of functions, that they should be familiar with theorems on convergence of series of orthogonal functions but not on convergence of martingales.

Convergence of sequences of measures is treated both in the general Vitali-Hahn-Saks setting and in the mathematical setting of Borel measures on the metric spaces of classical analysis: the compact metric spaces and the locally compact separable metric spaces. The general discussion is applied in detail to finite Lebesgue-Stieltjes measures on the line, in particular to probability measures.

Contents

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Conventions and Notation

1. Notation: Euclidean space

$\mathbf{R}^N$ denotes Euclidean N-space; $\mathbf{R} = \mathbf{R}^1$; $\mathbf{R}^+$ is the half line $[0,\infty)$; $\bar{\mathbf{R}}^+$ is the extended half-line $[0,+\infty]$; $\bar{\mathbf{R}}$ is the extended line $[-\infty,+\infty]$. The extended half-lines and lines can be metrized by giving them the metric of their images under the transformation $s' = \arctan s$.

2. Operations involving $\pm\infty$

$$a(\pm\infty) = \pm\infty \quad \text{if } a > 0, \qquad = 0 \quad \text{if } a = 0, \qquad = \mp\infty \quad \text{if } a < 0.$$

If a is finite, $a \pm \infty = \pm\infty$; if $a = +\infty$, $a + (+\infty) = +\infty$; if $a = -\infty$, $a + (-\infty) = -\infty$.

3. Inequalities and inclusions

"Positive" means "≥ 0"; "strictly positive" means "> 0." The symbols $\subset$ and $\supset$ allow equality. "Monotone" allows equality unless modified by "strictly." Thus the identically 0 function on $\mathbf{R}$ is both monotone increasing and decreasing, but is not strictly monotone in either direction.

4. A space and its subsets

If S is a space, the class of all its subsets is denoted by 2^S. The complement of a subset A of a space is denoted by $\tilde{A}$. If A and B are subsets of S, $\tilde{A} \cap B$ is sometimes denoted by $B - A$. The *indicator function* of a subset A of S, defined on S as 1 on A and 0 on $\tilde{A}$, is denoted by $\mathbf{1}_A$. In particular, the identically 1 function $\mathbf{1}_S$ will be denoted by $\mathbf{1}$ and the identically 0 function $\mathbf{1}_\varnothing$ by $\mathbf{0}$.

5. Notation: generation of classes of sets

If $\mathbf{A}$ is a class of subsets of a space, the classes $\mathbf{A}_\sigma$, $\mathbf{A}_\delta$, and $\tilde{\mathbf{A}}$ are, respectively, the classes of countable unions, countable intersections, and complements of the sets in $\mathbf{A}$.

6. Product sets

If $S_1,\ldots,S_n$ are sets, $S_1\times\cdots\times S_n$ is the *product set*

$$\{(s_1,\ldots,s_n): s_i \in S_i, (i \le n)\}.$$

If $\mathbf{A}_i$ is a class of subsets of S_i, $\mathbf{A}_1 \times\cdots\times \mathbf{A}_n$ is the class

$$\{A_1\times\cdots\times A_n: A_i \in \mathbf{A}_i\ (i \le n)\}$$

of product sets. The corresponding definitions are made for infinite (not necessarily countable) products.

7. Dot notation for an index set

"$B_\bullet$" is shorthand for $\{B_i, i \in I\}$, where I is a specified not necessarily countable index set. Unless the subscript range is otherwise described, "a finite sequence $B_\bullet$" means the sequence $B_1,\ldots,B_n$, for some strictly positive integer n, and "a sequence $B_\bullet$" means the infinite sequence $B_1,B_2,\ldots$. The notation $\Sigma B_\bullet$ means the sum over the values of the subscript, and corresponding dot notation will be applied to (not necessarily countable) set unions and intersections. If $a_\bullet$ is a sequence, the notation $\lim a_\bullet$ means $\lim_{n\to\infty} a_n$, and corresponding dot notation will be applied to inferior and superior limits. When dots appear more than once in an expression, the missing symbol is to be the same in each place. Thus if $A_\bullet$ and $B_\bullet$ are sequences of sets, $\cup(A_\bullet\cap B_\bullet)$ is the union of intersections $A_n\cap B_n$.

8. Notation: sets defined by conditions on functions

If f is a function from a space S into a space S' and if A' is a subset of S', the set notation $\{s \in S: f(s) \in A'\}$ will sometimes be abbreviated to $\{f \in A'\}$. Here f may represent a set of functions. Thus if $g_1,\ldots,g_n$ are functions from S into S' and if B' is a subset of S'^n, the notation $\{s \in S: [g_1(s),\ldots,g_n(s)] \in B'\}$ may be abbreviated to $\{(g_1,\ldots,g_n) \in A'\}$.

9. Notation: open and closed sets

The classes of open and closed subsets of a topological space will be denoted, respectively, by **G** and **F**.

10. Limit of a function at a point

The limit of a function at a point depends somewhat on the nationality and background of the writer. In this book, the limit does not involve the value of the function at the point. Thus the function $\mathbf{1}_{\{0\}}$, defined on **R** as 0 except at the origin, where the function is defined as 1, has limit 0 at the origin in this book even though the function does not have a Bourbaki limit at the origin.

11. Metric spaces

Recall that a *metric space* is a space coupled with a *metric*. A metric for a space S is a *distance function* d, a function from $S\times S$ into $\mathbf{R}^+$ satisfying the following conditions.

(a) Symmetry: $d(s,t) = d(t,s)$.
(b) Identity: $d(s,t) = 0$ if and only if $s = t$.
(c) Triangle inequality: $d(s,u) \le d(s,t) + d(t,u)$.

A *ball* in S is an open set $\{s\colon d(s,s_0) < r\}$; s_0 is the *center*, r is the *radius*.
It is a useful fact that if d is a metric for S and if c is a strictly positive constant, the function $d\wedge c$ is also a metric for S and determines the same topology as d. That is, the class of open sets is the same for $d\wedge c$ as for d. If d is a function from $S\times S$ into $\bar{\mathbf{R}}^+$ and satisfies (a), (b), and (c), the function $d\wedge c$ is a finite valued function satisfying these conditions and can therefore serve as a metric.

12. Standard metric space theorems

The following standard metric space theorems will be used. Proofs are sketched to facilitate checking by the reader that they are valid for the pseudometric spaces to be defined in Section 13.

(a) *A metric space* (S,d) *can be completed, that is, can be augmented by addition of new points to be complete.* To prove this theorem, let S' be the class of Cauchy sequences of points of S. The space S' is partitioned into equivalence classes, putting two Cauchy sequences $s_\bullet$ and $t_\bullet$ into the same equivalence class if and only if $\lim d(s_\bullet,t_\bullet) = 0$. If s' and t' are equivalence classes, define $d'(s',t') =$

$\lim d(s_\bullet, t_\bullet) = 0$. If s' and t' are equivalence classes, define $d'(s',t') = \lim d(s_\bullet, t_\bullet)$ for $s_\bullet$ in s' and $t_\bullet$ in t'. This limit exists, does not depend on the choice of Cauchy sequences in their equivalence classes, and (S',d') is a complete metric space. Define a function f from S into S' by $f(s) = s,s,s,\ldots$. This map preserves distance, and if S is identified with its image in S', S' is the desired completion of S.

(b) *A uniformly continuous function g from a dense subset of a metric space S into a complete metric space S' has a unique uniformly continuous extension to S.* To prove this theorem, observe that if s is not already in the domain of g, and if $s_\bullet$ is a sequence in the domain of g, with limit s, the uniform continuity of g implies that $\lim g(s_\bullet)$ exists and does not depend on the choice of $s_\bullet$. The value $g(s)$ is defined as this limit, and as so extended g is uniformly continuous on S. The uniqueness assertion is trivial.

(c) *If a complete metric space S is a countable union of closed sets, at least one summand has an inner point.* To prove this theorem, let $\cup S_\bullet$ be the union of a sequence of closed nowhere dense subsets of S. There is a closed ball B_1 of radius ≤ 1 in the open set $\tilde{S}_1$. Similarly there is a closed ball B_2 of radius $\leq 1/2$ in $B_1 \cap \tilde{S}_2$, and so on. The intersection of these closed balls is a point of S in no summand. Hence the union cannot be S, that is, if S is the union of a sequence of closed sets, at least one is not nowhere dense, and therefore has an inner point.

(d) *If $f_\bullet$ is a sequence of bounded continuous functions from a complete metric space S into* **R**, *and if* $\sup |f_\bullet(s)| < +\infty$ *for each point s of S, then there is a number γ for which the set* $\{s \in S\colon \sup |f_\bullet(s)| \leq \gamma\}$ *has an inner point.* This theorem follows at once from (c) because for each value of γ the set in question is closed, and as γ increases through the positive integers the set tends to S.

(e) A sequence $f_\bullet$ of functions from a metric space (S,d) into a metric space (S',d') is said to *converge uniformly at a point* s_0 of S, if there is convergence at s_0, and if to every strictly positive ε there corresponds a strictly positive δ and an integer k, with the property that $d'\big(f_m(s),f_n(s)\big) < \varepsilon$ whenever $n \geq k$, $m \geq k$, and $d(s,s_0) < \delta$. An equivalent condition is that there is a point s' of S' with the property that whenever $t_\bullet$ is a sequence in S, with limit s_0, then $\lim f_\bullet(t_\bullet) = s'$. *If $f_\bullet$ is a convergent sequence of continuous functions from S into S', the limit function f is continuous at every point of uniform convergence of the sequence.* In fact, if s_0 is a point of uniform convergence, if ε, δ, k are as just described, and if δ is decreased, if necessary, to make $d'\big(f_k(s),f_k(s_0)\big) < \varepsilon$ whenever $d(s,s_0) < \delta$, then

$$(12.1)\quad d'\big(f(s),f(s_0)\big) < d'\big(f(s),f_k(s)\big) + d'\big(f_k(s),f_k(s_0)\big) + d'\big(f_k(s_0),f(s_0)\big) < 3\varepsilon$$

whenever $d(s,s_0) < \delta$. Hence f is continuous at s_0, as asserted.

(f) *If a sequence $f_\bullet$ of continuous functions from a complete metric space (S,d) into a metric space (S',d') is convergent, there must be at least one point of uniform convergence.* (Since this assertion can be applied to the restrictions of the functions to an arbitrary closed ball in S, the set of points of uniform continuity of the sequence, and therefore the set of continuity points of the limit function, is actually dense in S.) This assertion is reduced to (c) as follows. For each pair of strictly positive integers m, k, the set