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Classical Geometries in Modern Contexts

Walter Benz

Geometry of Real Inner Product Spaces

Second Edition

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Preface

The basic structure playing the key role in this book is a *real inner product space* (X, δ), i.e. a real vector space X together with a mapping $\delta : X \times X \to \mathbb{R}$, a so-called *inner product*, satisfying rules (i), (ii), (iii), (iv) of section 1 of chapter 1. In order to avoid uninteresting cases from the point of view of geometry, we will assume throughout the whole book that there exist elements a, b in X which are linearly independent. But, on the other hand, we do *not* ask for the existence of a positive integer n such that every subset S of X containing exactly n elements is linearly dependent. In other words, we do *not* assume that X is a finite-dimensional vector space. So, when dealing in this book with different geometries like euclidean, hyperbolic, elliptic, spherical, Lorentz–Minkowskian geometry or Möbius (Lie) sphere geometry over a real inner product space (X, δ), the reader might think of $X = \mathbb{R}^2$ or $\mathbb{R}^3$, of X finite-dimensional, or of X infinite-dimensional. In fact, it plays no role, whatsoever, in our considerations whether the dimension of X is finite or infinite: the theory as presented does not depend on the dimension of X. In this sense we may say that our presentation in question is *dimension-free.*

The prerequisites for a fruitful reading of this book are essentially based on the sophomore level, especially after mastering basic linear algebra and basic geometry of $\mathbb{R}^2$ and $\mathbb{R}^3$. Of course, hyperspheres are defined via the inner product δ. At the same time we also define hyperplanes by this product, namely by $\{x \in X \mid \delta(a, x) = \alpha\}$, or, as we prefer to write $\{x \in X \mid ax = \alpha\}$, with $0 \neq a \in X$ and $\alpha \in \mathbb{R}$. This is a quite natural and simple definition and familiar to everybody who learned geometry, say, of the plane or of $\mathbb{R}^3$. For us it means that we do not need to speak about the existence of a basis of X (see, however, section 2.6 where we describe an example of a quasi-hyperplane which is not a hyperplane) and, furthermore, that we do not need to speak about (affine) hyperplanes as images under translations of maximal subspaces $\neq X$ of X (see R. Baer [1], p. 19), hence avoiding *transfinite* methods, which could be considered as somewhat strange in the context of geometries of Klein's Erlangen programme. This programme was published in 1872 by Felix Klein (1849–1925) under the title *Vergleichende Betrachtungen über neuere geometrische Forschungen, Programm zum Eintritt in die philosophische Facultät und den Senat der k. Friedrich-Alexander-Universität zu Erlangen* (Verlag von Andreas Deichert, Erlangen), and it gave rise to an ingenious and fundamental principle that allows distinguishing between different geometries (S, G) (see section 9 of chapter 1) on the basis of their groups G, their invariants and invariant notions (section 9). In connection with Klein's Erlangen programme compare also Julian Lowell Coolidge, A History of Geometrical Methods, Clarendon Press, Oxford, 1940, and, for instance, W. Benz [3], p. 38 f.

The papers [1] and [5] of E.M. Schröder must be considered as pioneer work for a dimension-free presentation of geometry. In [1], for instance, E.M. Schröder proved for arbitrary-dimensional $X, \dim X \geq 2$, that a mapping $f : X \to X$ satisfying $f(0) = 0$ and $\|x_1 - x_2\| = \|f(x_1) - f(x_2)\|$ for all $x_1, x_2 \in X$ with $\|x_1 - x_2\| = 1$ or 2 must be orthogonal. The methods of this result turned out to be important for certain other results of dimension-free geometry (see Theorem 4 of chapter 1 of the present book, see also W. Benz, H. Berens [1] or F. Radó, D. Andreescu, D. Válcan [1]).

The main result of chapter 1 is a common characterization of euclidean and hyperbolic geometry over (X, δ). With an implicit notion of a *(separable) translation group* T of X with axis $e \in X$ (see sections 7, 8 of chapter 1) the following theorem is proved (Theorem 7). Let d be a function, not identically zero, from $X \times X$ into the set $\mathbb{R}_{\geq 0}$ of all non-negative real numbers satisfying $d(x, y) = d(\varphi(x), \varphi(y))$ and, moreover, $d(\beta e, 0) = d(0, \beta e) = d(0, \alpha e) + d(\alpha e, \beta e)$ for all $x, y \in X$, all $\varphi \in T \cup O(X)$ where $O(X)$ is the group of orthogonal bijections of X, and for all real α, β with $0 \leq \alpha \leq \beta$. Then, up to isomorphism, there exist exactly two geometries with distance function d in question, namely the euclidean or the hyperbolic geometry over (X, δ). We would like to stress the fact that this result, the proof of which covers several pages, is also dimension-free, i.e. that it characterizes classical euclidean and classical (non-euclidean) hyperbolic geometry without restriction on the (finite or infinite) dimension of X, provided $\dim X \geq 2$. Hyperbolic geometry of the plane was discovered by J. Bolyai (1802–1860), C.F. Gauß (1777–1855), and N. Lobachevski (1793–1856) by denying the euclidean parallel axiom. In our Theorem 7 in question it is not a weakened axiom of *parallelity*, but a weakened notion of *translation with a fixed axis* which leads inescapably to euclidean or hyperbolic geometry and this for all dimensions of X with $\dim X \geq 2$. The methods of the proof of Theorem 7 depend heavily on the theory of functional equations. However, all results which are needed with respect to functional equations are proved in the book. Concerning monographs on functional equations see J. Aczél [1] and J. Aczél–J. Dhombres [1].

In chapter 2 the two metric spaces (X, eucl) (*euclidean metric space*) and (X, hyp) (*hyperbolic metric space*) are introduced depending on the different distance functions $\text{eucl}\,(x, y)$, $\text{hyp}\,(x, y)$ $(x, y \in X)$ of euclidean, hyperbolic geometry, respectively. The lines of these metric spaces are characterized in three different ways, as lines of L.M. Blumenthal (section 2), as lines of Karl Menger (section 3), or as follows (section 4): for given $a \neq b$ of X collect as *line through* a, b all $p \in X$ such that the system $d(a, p) = d(a, x)$ and $d(b, p) = d(b, x)$ of two equations has only the solution $x = p$. Moreover, subspaces of the metric spaces in question are defined in chapter 2, as well as spherical subspaces, parallelism, orthogonality, angles, measures of angles and, furthermore, with respect to $(X,\text{ hyp})$, equidistant surfaces, ends, horocycles, and angles of parallelism. As far as isometries of $(X,\text{ hyp})$ are concerned, we would like to mention the following main result (Theorem 35, chapter 2) which corresponds to Theorem 4 in chapter 1. Let $\varrho > 0$ be a fixed real number and $N > 1$ be a fixed integer. If $f : X \to X$ satisfies $\text{hyp}\,\big(f\,(x),\ f\,(y)\big) \leq \varrho$ for all $x, y \in X$ with $\text{hyp}\,(x,y) = \varrho$, and $\text{hyp}\,\big(f\,(x),\ f\,(y)\big) \geq N\varrho$ for all $x, y \in X$ with $\text{hyp}\,(x,y) = N\varrho$, then f must be an isometry of $(X,\text{ hyp})$, i.e. satisfies $\text{hyp}\,\big(f\,(x),\ f\,(y)\big) = \text{hyp}(x,y)$ for all $x, y \in X$. If the dimension of X is finite, the theorem of B. Farrahi [1] and A.V. Kuz'minyh [1] holds true: let $\varrho > 0$ be a fixed real number and $f : X \to X$ a mapping satisfying $\text{hyp}\,\big(f\,(x),\ f\,(y)\big) = \varrho$ for all $x, y \in X$ with $\text{hyp}\,(x,y) = \varrho$. Then f must already be an isometry. In section 21 of chapter 2 an example shows that this cannot generally be carried over to the infinite-dimensional case.

A geometry $\Gamma = (S, G)$ is a set $S \neq \emptyset$ together with a group G of bijections of S with the usual multiplication $(fg)(x) = f\,\big(g\,(x)\big)$ for all $x \in S$ and $f, g \in G$. The geometer then studies invariants and invariant notions of (S, G) (see section 9 of chapter 1). If a geometry Γ is based on an arbitrary real inner product space $X, \dim X \geq 2$, then it is useful, as we already realized before, to understand by "Γ, dimension-free" a theory of Γ which applies to every described X, no matter whether finite- or infinite-dimensional, so, for instance, the same way to $\mathbb{R}^2$ as to $C\,[0,1]$ with $fg = \int_0^1 t^2 f\,(t)\,g\,(t)\,dt$ for real-valued functions f, g defined and continuous in $[0,1]$ (see section 2, chapter 1). In chapter 3 we develop the geometry of Möbius dimension-free, and also the sphere geometry of Sophus Lie. Even Poincaré's model of hyperbolic geometry can be established dimension-free (see section 8 of chapter 3). In order to stress the fact that those and other theories are developed dimension-free, we avoided drawings in the book: drawings, of course, often present properly geometrical situations, but not, for instance, convincingly the ball $B\,(c,1)$ (see section 4 of chapter 2) of the above mentioned example with $X = C\,[0,1]$ such that $c : [0,1] \to \mathbb{R}$ is the function $c\,(\xi) = \xi^3$. The close connection between Lorentz transformations (see section 17 of chapter 3) and Lie transformations (section 12), more precisely Laguerre transformations (section 13), has been known for almost a hundred years: it was discovered by H. Bateman [1] and H.E. Timerding [1], of course, in the classical context of four dimensions (section 17). This close connection can also be established dimension-free, as shown in chapter 3. A fundamental theorem in Lorentz–Minkowski geometry (see section 17, chapter 3) of A.D. Alexandrov [1] must be mentioned here with respect to Lie sphere geometry: if $(2 \leq) \dim X < \infty$, and if $\lambda : Z \to Z,\ Z := X \oplus \mathbb{R}$, is a bijection such that the Lorentz–Minkowski distance $l\,(x,y)$ (section 1 of chapter 4) is zero if, and only if $l\,\big(f\,(x),\ f\,(y)\big) = 0$ for all $x, y \in Z$, then f is a Lorentz transformation up to a dilatation. In fact, much more than this follows from Theorem 65 (section 17, chapter 3) which is a theorem of Lie (Laguerre) geometry: we obtain from Theorem 65 Alexandrov's theorem in the dimension-free version and this even in the Cacciafesta form (Cacciafesta [1]) (see Theorem 2 of chapter 4).

All Lorentz transformations of Lorentz–Minkowski geometry over (X, δ) are determined dimension-free in chapter 4, section 1, by Lorentz boosts (section 14, chapter 3), orthogonal mappings and translations. Also this result follows from a theorem (Theorem 61 in section 14, chapter 3) on Lie transformations. In Theorem 6 (section 2, chapter 4) we prove dimension-free a well-known theorem of Alexandrov–Ovchinnikova–Zeeman which these authors have shown under the assumption $\dim X < \infty$, and in which all causal automorphisms (section 2, chapter 4) of Lorentz–Minkowski geometry over (X, δ) are determined.

In sections 9, 10, 11 (chapter 4) Einstein's cylindrical world over (X, δ) is introduced and studied dimension-free; moreover, in sections 12, 13 we discuss de Sitter's world. Sections 14, 15, 16, 17, 18, 19 are devoted to elliptic and spherical geometry. They are studied dimension-free as well. In section 19 the classical lines of spherical, elliptic geometry, respectively, are characterized via functional equations. The notions of Lorentz boost and hyperbolic translation are closely connected: this will be proved and discussed in section 20, again dimension-free.

It is a pleasant task for an author to thank those who have helped him. I am deeply thankful to Alice Günther who provided me with many valuable suggestions on the preparation of this book. Furthermore, the manuscript was critically revised by my colleague Jens Schwaiger from the university of Graz, Austria. He supplied me with an extensive list of suggestions and corrections which led to substantial improvements in my exposition. It is with pleasure that I express my gratitude to him for all the time and energy he has spent on my work.

Waterloo, Ontario, Canada, June 2005 Walter Benz

Preface to the Second Edition

In this second edition a new chapter (δ–Projective Mappings, Isomorphism Theorems) was added. One of the fundamental results contained in this chapter 5 is that the hyperbolic geometries over two (not necessarily finite–dimensional) real inner product spaces (X, δ), (V, ε) (see p. 1) are isomorphic (p. 16f) if, and only if, the two underlying real inner product spaces are isomorphic (p. 1f) as well. Similar theorems are proved for Möbius sphere geometries and for the euclidean case. Another result of chapter 5 we would like to mention is that the Cayley–Klein model of hyperbolic geometry over (X, δ), as developed dimension–free in section 2.12, can also be established dimension–free via a certain selection of projective mappings of X depending, however, on the chosen inner product δ of X.

It remains to the author to thank Professors Hans Havlicek, Zsolt Páles, Victor Pambuccian who, through their support, their criticism and their suggestions, contributed to the improvement of this book. Special thanks in this connection are due to Alice Günther and my colleagues Ludwig Reich and Jens Schwaiger.

Last, but not least, I would like to express my gratitude to the Birkhäuser publishing company and, especially, to Dr. Thomas Hempfling for their conscientious work and helpful cooperation.

Hamburg, July 2007 Walter Benz

Contents

Chapter 1

Translation Groups

1.1 Real inner product spaces

A *real inner product space* (X, δ) is a real vector space X together with a mapping $\delta : X \times X \rightarrow \mathbb{R}$ satisfying

(i) $\delta(x, y) = \delta(y, x)$,

(ii) $\delta(x + y, z) = \delta(x, z) + \delta(y, z)$,

(iii) $\delta(\lambda x, y) = \lambda \cdot \delta(x, y)$,

(iv) $\delta(x, x) > 0$ for $x \neq 0$

for all $x, y, z \in X$ and $\lambda \in \mathbb{R}$. Concerning the notation $\delta : X \times X \rightarrow \mathbb{R}$ and others we shall use later on, see the section *Notation and symbols* of this book. Instead of $\delta(x, y)$ we will write xy or, occasionally, $x \cdot y$. The laws above are then the following:

$$xy = yx,\ (x + y) z = xz + yz,\ (\lambda x) \cdot y = \lambda \cdot (xy)$$

for all $x, y, z \in X$, $\lambda \in \mathbb{R}$, and $x^2 := x \cdot x > 0$ for all $x \in X \backslash \{0\}$. Instead of (X, δ) we mostly will speak of X, hence tacitly assuming that X is equipped with a fixed *inner product*, i.e. with a fixed $\delta : X \times X \rightarrow \mathbb{R}$ satisfying rules (i), (ii), (iii), (iv).

Two real inner product spaces (X, δ), (X', δ') are called *isomorphic* provided (in the sense of *if, and only if*) there exists a bijection

$$\varphi : X \rightarrow X'$$

such that

$$\varphi(x + y) = \varphi(x) + \varphi(y),\ \varphi(\lambda x) = \lambda \varphi(x),\ \delta(x, y) = \delta'(\varphi(x), \varphi(y))$$

hold true for all $x, y \in X$ and $\lambda \in \mathbb{R}$. The last of these equations can be replaced by the weaker one $\delta(x, x) = \delta'\big(\varphi(x), \varphi(x)\big)$ for all $x \in X$, since

$$2\overline{\delta}(x, y) = \overline{\delta}(x + y, x + y) - \overline{\delta}(x, x) - \overline{\delta}(y, y)$$

holds true for all $x, y \in X$ for $\overline{\delta} = \delta$ as well as for all $x, y \in X'$ for $\overline{\delta} = \delta'$.

1.2 Examples

a) Let $B \neq \emptyset$ be a set and define $X(B)$ to be the set of all $f : B \to \mathbb{R}$ such that $\{b \in B \mid f(b) \neq 0\}$ is finite. Put

$$(f + g)(b) := f(b) + g(b)$$

for $f, g \in X$ and $b \in B$, and

$$(\alpha f)(b) := \alpha f(b)$$

for $f \in X$, $\alpha \in \mathbb{R}$, $b \in B$. Finally set

$$fg := \sum_{b \in B} f(b)\, g(b)$$

for $f, g \in X$.

b) Let $\alpha < \beta$ be real numbers and let X be the set of all continuous functions $f : [\alpha, \beta] \to \mathbb{R}$ with $[\alpha, \beta] := \{t \in \mathbb{R} \mid \alpha \leq t \leq \beta\}$. Define $f + g$, αf as in a) and put

$$fg := \int_\alpha^\beta h(t)\, f(t)\, g(t)\, dt$$

for a fixed $h \in X$ satisfying $h(t) > 0$ for all $t \in [\alpha, \beta] \backslash T$ where T is a finite subset of $[\alpha, \beta]$. This real inner product space will be denoted by $X\big([\alpha, \beta], h\big)$.

c) Suppose that X is the set of all sequences

$$(a_1, a_2, a_3, \ldots)$$

of real numbers $a_1, a_2, a_3, \ldots$ such that $\sum_{i=1}^\infty a_i^2$ exists. Define

$$\begin{aligned}
(a_1, a_2, \ldots) + (b_1, b_2, \ldots) &:= (a_1 + b_1, a_2 + b_2, \ldots), \\
\lambda \cdot (a_1, a_2, \ldots) &:= (\lambda a_1, \lambda a_2, \ldots), \\
(a_1, a_2, \ldots) \cdot (b_1, b_2, \ldots) &:= \sum_{i=1}^\infty a_i b_i,
\end{aligned}$$

by observing

$$(a_i + b_i)^2 = a_i^2 + b_i^2 + 2a_i b_i \leq a_i^2 + b_i^2 + a_i^2 + b_i^2$$

from $(a_i - b_i)^2 \geq 0$, i.e. by noticing

$$\sum_{i=1}^{n}(a_i + b_i)^2 \leq 2\sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} b_i^2,$$

i.e. that $\sum_{i=1}^{\infty}(a_i + b_i)^2$ exists. Because of

$$4\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n}(a_i + b_i)^2 - \sum_{i=1}^{n}(a_i - b_i)^2,$$

also $\sum_{i=1}^{\infty} a_i b_i$ exists.

1.3 Isomorphic, non-isomorphic spaces

Let n be a positive integer. The $\mathbb{R}^n$ consists of all ordered n-tuples

$$(x_1, x_2, \ldots, x_n)$$

of real numbers x_i, $i = 1, 2, \ldots, n$. It is a real inner product space with

$$\begin{aligned}
(x_1, \ldots, x_n) + (y_1, \ldots, y_n) &:= (x_1 + y_1, \ldots, x_n + y_n), \\
\alpha \cdot (x_1, \ldots, x_n) &:= (\alpha x_1, \ldots, \alpha x_n), \\
(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) &:= x_1 y_1 + \cdots + x_n y_n
\end{aligned}$$

for $x_i, y_i, \alpha \in \mathbb{R}$, $i = 1, \ldots, n$.

Obviously, $\mathbb{R}^n$ and $X\big(\{1, 2, \ldots, n\}\big)$ are isomorphic: define $\varphi(x_1, \ldots, x_n)$ to be the function $f : \{1, \ldots, n\} \to \mathbb{R}$ with $f(i) = x_i$, $i = 1, \ldots, n$.

Suppose that B_1, B_2 are non-empty sets. The real inner product spaces $X(B_1)$, $X(B_2)$ are isomorphic if, and only if, there exists a bijection $\gamma : B_1 \to B_2$ between B_1 and B_2. If there exists such a bijection, define $\varphi(f)$ for $f \in X(B_1)$ by

$$\varphi(f)\big(\gamma(b)\big) = f(b)$$

for all $b \in B_1$. Hence $\varphi : X(B_1) \to X(B_2)$ establishes an isomorphism. If $X(B_1)$, $X(B_2)$ are isomorphic, there exists a bijection

$$\varphi : X(B_1) \to X(B_2)$$

with $\varphi(x + y) = \varphi(x) + \varphi(y)$, $\varphi(\lambda x) = \lambda\varphi(x)$ for all $x, y \in X(B_1)$ and $\lambda \in \mathbb{R}$. We associate to $b \in B_1$ the element $\hat{b}$ of $X(B_1)$ defined by $\hat{b}(b) = 1$ and $\hat{b}(c) = 0$ for all $c \in B_1 \backslash \{b\}$. Then $\hat{B}_1 := \{\hat{b} \mid b \in B_1\}$ is a basis of $X(B_1)$, and $\hat{B}_2$ and $\varphi(\hat{B}_1)$ must be bases of $X(B_2)$. Since $\hat{B}_1$, $\varphi(\hat{B}_1)$ are of the same cardinality, and also $\hat{B}_2$, $\varphi(\hat{B}_1)$, we get the same cardinality for $\hat{B}_1, \hat{B}_2$, and hence also for B_1, B_2.

Suppose that $\alpha < \beta$ are real numbers and that $h : [\alpha, \beta] \to \mathbb{R}$ is continuous with $h(\eta) > 0$ in $[\alpha, \beta]$. Then the real inner product spaces $X\big([\alpha, \beta], h\big)$ and $X\big([0,1], 1\big)$ are isomorphic. Here 1 designates the function $1(\xi) = 1$ for all $\xi \in [0,1]$. In order to prove this statement, associate to the function $f : [0,1] \to \mathbb{R}$, also written as $f(\xi)$, the function $\varphi(f) : [\alpha, \beta] \to \mathbb{R}$ defined by

$$\varphi(f)(\eta) := \sqrt{\frac{(\beta-\alpha)^{-1}}{h(\eta)}}\; f\left(\frac{\eta-\alpha}{\beta-\alpha}\right).$$

Obviously, $\varphi : X\big([0,1], 1\big) \to X\big([\alpha, \beta], h\big)$ is a bijection. It satisfies

$$\varphi(f+g) = \varphi(f) + \varphi(g),\ \ \varphi(\lambda f) = \lambda \varphi(f)$$

for all $\lambda \in \mathbb{R}$ and $f, g \in X\big([0,1], 1\big)$. Moreover, we obtain

$$\varphi(f) \cdot \varphi(g) = \int_\alpha^\beta h(\eta)\, \varphi(f)(\eta)\, \varphi(g)(\eta)\, d\eta = \int_0^1 f(\xi)\, g(\xi)\, d\xi = f \cdot g,$$

and hence that $X\big([0,1], 1\big)$, $X\big([\alpha, \beta], h\big)$ are isomorphic.

Remark. There exist examples of (necessarily infinite-dimensional) real vector spaces X with mappings $\delta_\nu : X \times X \to \mathbb{R}$, $\nu = 1, 2$, satisfying rules (i),(ii), (iii), (iv) of section 1.1 such that (X, δ_1) and (X, δ_2) are not isomorphic (J. Rätz [1]).

1.4 Inequality of Cauchy–Schwarz

Inequality of Cauchy–Schwarz: If a, b are elements of X, then $(ab)^2 \le a^2 b^2$ holds true.

Proof. Case $b = 0$. Observe, for $p \in X$,

$$pb = p \cdot 0 = p \cdot (0 + 0) = p \cdot 0 + p \cdot 0,$$

i.e. $pb = p \cdot 0 = 0$, i.e. $a \cdot b = 0$ and $b^2 = 0$.

Case $b \neq 0$. Hence $b^2 > 0$ and thus

$$0 \le \left(a - \frac{ab}{b^2}\, b\right)^2 = a^2 - \frac{(ab)^2}{b^2}, \tag{1.1}$$

i.e. $(ab)^2 \le a^2 b^2$. □

Lemma 1. *If a, b are elements of X such that $(ab)^2 = a^2 b^2$ holds true, then a, b are linearly dependent.*

Proof. Case $b = 0$. Here $0 \cdot a + 1 \cdot b = 0$.

Case $b \neq 0$. Hence, by (1.1), $\left(a - \frac{ab}{b^2} \cdot b\right)^2 = 0$, i.e.

$$a - \frac{ab}{b^2} \cdot b = 0. \qquad \square$$

For $x \in X$, the real number $s \geq 0$ with $s^2 = x^2$ is said to be the *norm* of x, $s =: \|x\|$. Obviously, $\|\lambda x\| = |\lambda| \cdot \|x\|$ for $\lambda \in \mathbb{R}$ and $x \in X$. Moreover, $\|x\| = 0$ holds true for $x \in X$ if, and only if, $x = 0$. Observing $xy \leq |xy| \leq \|x\| \cdot \|y\|$ for $x, y \in X$, from the inequality of Cauchy–Schwarz, we obtain $(x+y)^2 \leq \left(\|x\| + \|y\|\right)^2$, i.e. we get the *triangle inequality*

$$\|x + y\| \leq \|x\| + \|y\| \text{ for all } x, y \in X. \tag{1.2}$$

1.5 Orthogonal mappings

Let X be a real inner product space. In order to avoid that the underlying real vector space of X is $\mathbb{R}$ or $\{0\}$, *we will assume throughout the whole book that there exist two elements in X which are linearly independent.* Under this assumption the following holds true: if *x, y are elements of X, there exists $w \in X$ with $w^2 = 1$ and $w \cdot (x - y) = 0$.* Since there are elements a, b in X, which are linearly independent, put $w = \frac{a}{\|a\|}$ in the case $x = y$. If $x \neq y$, there exists z in X such that $z \notin \mathbb{R} \cdot (x-y)$, because otherwise $a, b \in \mathbb{R} \cdot (x - y)$ would be linearly dependent. Hence

$$v := z - \frac{z\,(x-y)}{(x-y)^2}\,(x-y) \neq 0.$$

Thus $w := \frac{v}{\|v\|}$ satisfies $w^2 = 1$ and $w \cdot (x - y) = 0$.

A mapping $\omega : X \to X$ is called *orthogonal* if, and only if,

$$\omega\,(x + y) = \omega\,(x) + \omega\,(y),\ \omega\,(\lambda x) = \lambda \omega\,(x),\ xy = \omega\,(x)\,\omega\,(y)$$

hold true for all $x, y \in X$ and $\lambda \in \mathbb{R}$.

An orthogonal mapping ω of X must be injective, but it need not be surjective. Assume $\omega\,(x) = \omega\,(y)$ for the elements x, y of X. Because of

$$\omega\,(x - y) = \omega\left(x + [(-1)\,y]\right) = \omega\,(x) + (-1)\,\omega\,(y) = 0,$$

we obtain $(x - y)^2 = [\omega\,(x - y)]^2 = 0$, i.e. $x - y = 0$, i.e. $x = y$.

Define $B := \{1, 2, 3, \ldots\}$ and take the space $X = X\,(B)$ of type a). For $f \in X$ put

$$\omega\,(f)(1) = 0 \text{ and } \omega\,(f)(i) = f\,(i-1),\ i = 2, 3, \ldots.$$