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Nonlinear Elliptic Equations and Nonassociative Algebras

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Preface

The present volume contains some applications of noncommutative and nonassociative algebras to constructing unusual (nonclassical and singular) solutions to fully nonlinear elliptic partial differential equations of second order. Here the solutions are to be understood in a weak (viscosity) sense. Using such algebras to construct exotic or specific analytic and geometric structures is not new. One can mention here, for instance, the constructions of exotic spheres by Milnor [**163**], of singular solutions to the minimal surface system by Lawson and Osserman [**144**], the ADHM and constuction of instantons by Atiyah, Drinfel'd, Hitchin, and Manin [**17**], and the construction of singular coassociative manifolds by Harvey and Lawson [**103**], all four using quaternions, as well as the recent constructions of unusual solutions of the Ginzburg-Landau system by Farina and Ge-Xie [**86**], [**93**], using isoparametric polynomials and thus, implicitly, Jordan or Clifford algebras.

However, our applications of quaternions, octonions, and Jordan algebras to elliptic partial differential equations of second order are new; they allow us to solve a longstanding problem of the existence of truly weak viscosity solutions, which are not smooth (= classical) ones. Moreover, in some sense, they give (albeit along with some other arguments) an almost complete description of homogeneous solutions to fully nonlinear elliptic equations. In fact, a major part of the book is devoted to the simplest class of fully nonlinear uniformly elliptic equations, namely those of the form

$$F(D^2u) = 0, \tag{0.1}$$

F being a nonlinear sufficiently smooth functional on symmetric matrices and D^2u being the Hessian of a putative solution u. Those are "constant coefficient" fully nonlinear elliptic equations. Moreover, often we impose a rather drastic condition that F depends only on the eigenvalues of the Hessian (so-called "Hessian equations"). In that case F is a function of only n values of symmetric functions of D^2u rather than of $n(n+1)/2$ partial derivatives, n being the dimension of the ambient space. Our methods show that even in that very restricted setting in five and more dimensions (some of) those equations admit homogeneous δ-order solutions with any $\delta \in]1, 2]$, that is, of all orders compatible with known regularity results by Caffarelli and Trudinger [**37**], [**255**] for viscosity solutions of fully nonlinear uniformly elliptic equations, proving the optimality of these regularity results. To the contrary, the situation in four and fewer dimensions is completely different. First of all, in two dimensions the classical result by L. Nirenberg [**186**] guarantees the regularity of all viscosity solutions, homogeneous or not. In Section 1.6 we prove that in four (and thus three) dimensions there are no homogeneous order 2 solutions to fully nonlinear uniformly elliptic equations, at least in the analytic setting, which suggests strongly that there are no nonclassical homogeneous solutions at all in four and three dimensions. If so, we get a complete list of dimensions where nonclassical homogeneous solutions to fully nonlinear uniformly elliptic equations do exist. One can compare this with the situation of, say, ten years ago, when the very existence of nonclassical viscosity solutions was not known.

We should repeat once more that this result of fundamental importance for the theory of partial differential equations is obtained by applications of relatively elementary algebraic (and differential geometric) means, thus stressing once more that studying relations between apparently disconnected mathematical areas can often be very fruitful.

Furthermore, there are some cases where (singular) solutions of some classes of nonlinear elliptic equations and some nonassociative algebras are interrelated even more strongly, leading in certain circumstances to the equivalence of those objects. A study of these relations and their applications to classifying both classes of objects is the second theme of the book, intimately related to the previous one.

Our exposition is as follows. Since we hope that our work can be of use to a rather diversified mathematical audience, we devote the first three chapters to the basics of nonlinear elliptic equations and of noncommutative and nonassociative algebraic structures used in our constructions.

In Chapter 1 we recall basic facts about nonlinear elliptic equations and their viscosity solutions. The material in the first five sections is quite traditional in many papers devoted to viscosity solutions. However, in Section 1.2 we also formulate two recent results on partial regularity of solutions, in Section 1.3 we expose a recent result concerning the difference of viscosity solutions, and in Section 1.4 we give a recent result on the regularity of solutions to axially symmetric Dirichlet problems for Hessian equations. Section 1.6 is devoted to recent results and conjectures for homogeneous solutions to fully nonlinear uniformly elliptic equations. Section 1.7 gives some Liouville type results and various results on removable singularities for solutions of fully nonlinear elliptic equations, including a recent result describing viscosity solutions of a uniformly elliptic Hessian equation in a punctured ball.

Chapter 2 is devoted to the construction and elementary properties of the real division algebras $\mathbb{H}, \mathbb{O}$, Clifford algebras, spinor groups, and some exceptional Lee groups, especially G_2. We also discuss cross products in the algebra $\mathbb{O}$ and the resulting calibrations (in their algebraic form).

In Chapter 3 we give an overview of Jordan algebras in their relation to special cubics and some partial differential equations (of first order). Most of its material is classical, but some new facts concerning relations between cubic Jordan algebras and the so-called eiconal differential equation, $|\nabla f(x)|^2 = c|x|^4$, are proven.

In Chapters 4 and 5 we give our main constructions of nonclassical and singular solutions to fully nonlinear, uniformly elliptic equations, often of Hessian or of Isaacs type. In fact, all our nonclassical solutions are of the form $P(x)/|x|^\alpha$ with a homogeneous polynomial $P(x), x \in \mathbb{R}^n$, of degree 3, 4, or 6 and a suitable α. Chapter 4 contains the constructions based on trialities, which use real division algebras: quaternions and octonions; there $n = 12$ or 24, $\deg P = 3$, $\alpha \in [1, 2[$. Chapter 5 gives constructions based on isoparametric polynomials $P(x)$, $x \in \mathbb{R}^n$, $n \geq 5$, of degrees 3, 4, or 6 coming from Jordan and Clifford algebras. The constructions of Chapter 4 are more elementary in that they use less of algebraic theory but need more calculations than those of Chapter 5. The arguments in these chapters are based on several closely related criteria for solutions of fully nonlinear uniformly elliptic equations in terms of appropriate combinations of the spectrum for their Hessians. Those conditions are extremely restrictive and one needs rather elaborated arguments and/or calculations to verify them, which is obtained partially by using MAPLE calculations. One notes, however, that the calculations in Chapters 4 and 5 (and in Chapter 7) which use MAPLE extensively are completely rigorous since there MAPLE is used to verify algebraic identities, albeit rather cumbersome ones.

Chapter 6 is devoted to a classification of cubic minimal cones, that is, the simplest nontrivial solutions to the minimal surface system which is (almost) complete under a natural additional condition (the case of radial eigencubics). The main method there is to construct a certain nonassociative algebra from a given minimal cubic cone in such a way that the differential-analytical structure of the cones becomes transparent from the algebraic side, and vice versa. The main tool for this is the so-called Freudenthal multiplication. It associates to any fixed cubic form u on a vector space V carrying a symmetric nondegenerate bilinear form Q the multiplication $(x, y) \to xy$ by setting $\partial_x \partial_y u|_z = Q(xy; z)$. The algebra $V(u)$ defined in this way is called the Freudenthal-Springer algebra of the cubic form u. In the basic case of a radial eigencubic the corresponding Freudenthal-Springer algebra leads to a so-called radial eigencubic algebra, or just a *REC algebra*. Thus, the classification of radial eigencubics becomes equivalent to that of REC algebras. There exist two principal classes of REC algebras, namely those coming from Clifford and Jordan structures, respectively. Applying standard methods of nonassociative algebra such as Pierce decomposition and a thorough study of certain defining relations in REC algebras, one eventually gets their complete classification. Note, however, that the algebraic techniques of this chapter are elaborated more than in the other chapters and assume more advanced knowledge of the nonassociative algebraic systems.

In Chapter 7 we treat elliptic equations arising in calibrated geometry [**103**], namely, the special Lagrangian, associative, coassociative, and Cayley equations; they are not uniformly, but only strictly, elliptic, and we recall briefly their constructions in Section 7.1. One notes, however, that the construction of singular coassociative 4-folds given by Harvey and Lawson in [**103**] and recalled in Section 7.2 resembles strongly the constructions in Chapter 4. It would be very interesting to understand a possible common ground of constructions in Chapters 4 and 7 (and, presumably, in Chapters 5 and 6) and eventually find some other situations where it works. Sections 7.3 and 7.4 are devoted to constructions of some singular solutions to the special Lagrangian equations (SLE) in the nonconvex case, in three dimensions. Note that in the convex (or concave) case those solutions are smooth in any dimension by [**48**] and that in two dimensions these equations cannot be nonconvex. These constructions also lead to examples of a failure of the maximum principle for the Hessian of solutions to a uniformly elliptic equation in three and more dimensions as well as to examples of solutions to the minimal surface system with a notably low regularity.

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Contents

CHAPTER 1

Nonlinear Elliptic Equations

In this chapter we give a brief introduction to the theory of second-order elliptic equations, principally fully nonlinear ones. After defining them in Section 1.1 and giving some basic examples, we formulate the principal problems concerning the solutions of such equations, namely, their existence, uniqueness, and regularity. The main properties of the viscosity (weak) solutions giving a partial answer to these problems are discussed in Section 1.2. In addition to the now classical foundational results, we formulate there two recent results on partial regularity of solutions, namely Theorems 1.2.5 and 1.2.6. In Section 1.3 we consider linear elliptic operators of nondivergence form; in addition to classical results, we expose a recent result concerning the difference of viscosity solutions, Theorem 1.3.6. Section 1.4 is devoted to nonlinear equations with smooth solutions, i.e., those with a convex functional F; we describe the Evans-Krylov theory, which guarantees that under the convexity assumption the viscosity solutions are classical, i.e., really verify the equation. Note that one of our principal aims in this book is to show that without the convexity assumption this regularity *does not* hold. We also give a recent result, Theorem 1.4.4, which guarantees the same property for axially symmetric Dirichlet problems for Hessian equations. Section 1.5 is devoted to degenerate elliptic equations; first we expose a geometric approach to degenerate elliptic equations proposed recently by Harvey and Lawson [**104**] and then discuss various regularity results for them. In Section 1.6 we formulate some results and conjectures for homogeneous solutions to fully nonlinear uniformly elliptic equations. Finally, in Section 1.7 we give some Liouville type theorems and also various results on removable singularities for solutions of fully nonlinear elliptic equations, including a recent result, Theorem 1.7.6, describing viscosity solutions of a uniformly elliptic Hessian equation in a punctured ball.

The literature devoted to the topic is overwhelming; the basic references concerning Sections 1.1–1.4 are [**94**], [**42**], [**71**], [**133**], [**134**]; see also [**47**], [**46**], [**48**], [**83**], [**137**], [**37**], [**253**], [**254**], [**255**], [**36**].

1.1. Elliptic equations

1.1.1. Definition and examples. Throughout this book we consider second-order partial differential equations of the form

$$F(D^2u, Du, u, x) = 0. \tag{1.1.1}$$

Here D^2u denotes the Hessian of the function $u : \Omega \longrightarrow \mathbb{R}$, Du being its gradient, $x \in \Omega \subset \mathbb{R}^n$ for a domain (= open connected set) Ω. The functional is of the form

$$F = F(X, r, p, x) : A \times \mathbb{R}^n \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$$

where A is a domain in $\mathrm{Sym}_n(\mathbb{R})$, the space of symmetric real $n \times n$-matrices over $\mathbb{R}$. The degenerate ellipticity condition is given by

$$F(X, r, p, x) \leq F(Y, r, p, x) \text{ if } X \leq Y, \tag{1.1.2}$$

i.e., the matrix $Y - X$ is nonnegatively defined; we also suppose that F is continuous. Often we will demand more on the functional F.

If F is a C^1-function in X, (1.1.2) yields the inequality

$$F_X \geq 0,$$

i.e., the matrix of the first derivatives of F with respect to the X variables is nonnegative. If in addition $A = \mathrm{Sym}_n(\mathbb{R})$, the last two inequalities are equivalent.

Let us give several well-known examples of elliptic equations which describe important natural processes, geometrical problems, and the like.

EXAMPLE 1.1.1 (Laplace's equation).

$$\Delta u - c(x)u = f(x).$$

The corresponding F is given by $F(X; p; r; x) = \mathrm{tr}(X) - c(x)r + f(x)$. As particular cases we get the classical Laplace equation $\Delta u = 0$ defining harmonic functions and the Poisson equation $\Delta u = f(x)$.

EXAMPLE 1.1.2 (Degenerate elliptic linear equations). Degenerate elliptic linear equations are of the form

$$\sum a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum b_i(x) \frac{\partial u}{\partial x_i} - c(x)u(x) = f(x),$$

where the matrix $A(x) = a_{ij}(x)$ is symmetric; the corresponding F is

$$F(X; p; r; x) = \mathrm{tr}(A(x)X) - \sum b_i(x)p_i - c(x)r - f(x).$$

In this case, F is degenerate elliptic if and only if $A(x) \geq 0$. If a constant $C > 0$ exists such that $CI \geq A(x) \geq C^{-1}I$ for all $x \in \Omega$ where I is the identity matrix, F is said to be *uniformly elliptic*. If $C(x)I \geq A(x) \geq C(x)^{-1}I$ for $C(x) > 0$ and any $x \in \Omega$, F is called *strictly elliptic*.

EXAMPLE 1.1.3 (Quasilinear elliptic equations in divergence form). An equation

$$\sum \frac{\partial}{\partial x_i}(a_i(Du, x)) - b(Du, u, x) = 0 \tag{1.1.3}$$

is elliptic if the vector field $a(p, x)$ is monotone in p regarded as a mapping from $\mathbb{R}^n$ to itself. If the coefficients are differentiable, one can rewrite (1.1.3) as (1.1.1) with

$$F(X, p, r, x) = \mathrm{tr}((D_p a(p, x))X) - b(p, r, x) + \sum \frac{\partial a_i(p, x)}{\partial x_i}.$$

EXAMPLE 1.1.4 (p-Laplace equation). Let $p \geq 1$. The equation

$$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u) = 0 \tag{1.1.4}$$

is an important example of a quasilinear elliptic equation in divergence form; one easily calculates that

$$\Delta_p u = |\nabla u|^{p-4} \left\{ |\nabla u|^2 \Delta u + (p-2) \sum_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \right\}.$$

For $p = 1$ one gets

$$-\Delta_1 u = -\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = H$$

where H is the mean curvature operator. For $p = 2$ one returns to the Laplacian: $\Delta_2 u = \Delta u$.

When $p = n$ is the dimension of the ambient space, the operator Δ_n becomes conformally invariant.

For $p = \infty$ one gets the limit ∞-*Laplacian,*

$$\Delta_\infty u = \sum_{i,j} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

EXAMPLE 1.1.5 (Quasilinear elliptic equations in nondivergence form). The equation

$$\sum a_{ij}(p,x)\frac{\partial^2 u}{\partial x_i \partial x_j} - b(Du, u, x) = 0,$$

where $A(p,x) = a_{ij}(p,x) \in \operatorname{Sym}_n(\mathbb{R})$, corresponds to

$$F(X, r, p, x) = \operatorname{tr}(A(p,x)X) - b(p, r, x).$$

EXAMPLE 1.1.6 (Pucci's equations). These are uniformly elliptic equations important in many applications, especially in the theory of viscosity solutions for fully nonlinear elliptic equations. They are of the form

$$\mathcal{M}^-(D^2u) = f(x), \quad \mathcal{M}^+(D^2u) = f(x)$$

where $\mathcal{M}^-$ and $\mathcal{M}^+$ are *Pucci's extremal operators* defined as follows.

Let $A \in \operatorname{Sym}_n(\mathbb{R})$ and let $\lambda \in]0, \Lambda]$. Define

$$\mathcal{M}^-(A, \lambda, \Lambda) = \mathcal{M}^-(A) = \lambda \sum_{\lambda_i > 0} \lambda_i + \Lambda \sum_{\lambda_i < 0} \lambda_i,$$
$$\mathcal{M}^+(A, \lambda, \Lambda) = \mathcal{M}^+(A) = \lambda \sum_{\lambda_i < 0} \lambda_i + \Lambda \sum_{\lambda_i > 0} \lambda_i,$$

where λ_i, $i = 1, 2, \ldots, n$, are the eigenvalues of A.

An equivalent definition is given by

$$\mathcal{M}^-(A, \lambda, \Lambda) = \inf_{M \in M_{\lambda,\Lambda}} \operatorname{tr}(MA), \quad \mathcal{M}^+(A, \lambda, \Lambda) = \sup_{M \in M_{\lambda,\Lambda}} \operatorname{tr}(MA),$$

$M_{\lambda,\Lambda}$ being the set of all symmetric matrices with all eigenvalues in $[\lambda, \Lambda]$.

One verifies without difficulty that $\mathcal{M}^-$, $\mathcal{M}^+$ are uniformly elliptic with the ellipticity constant $C := \max\{n\lambda, n\Lambda\}$ and that $\mathcal{M}^-$ is concave and $\mathcal{M}^+$ is convex.

EXAMPLE 1.1.7 (Hamilton-Jacobi-Bellman and Isaacs equations). These are the fundamental partial differential equations for stochastic control and stochastic differential games. The natural setting involves a collection of elliptic operators of second-order depending either on one parameter α (in the Hamilton-Jacobi-Bellman case) or two parameters α, β (in the case of Isaacs equations). These parameters lie in some index sets.

Let us define for $a_{ij}^\alpha(x), a_{ij}^{\alpha,\beta}(x) \in \operatorname{Sym}_n(\mathbb{R})$

$$\mathcal{L}^\alpha u := \sum a_{ij}^\alpha(x)\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum b_i^\alpha(x)\frac{\partial u}{\partial x_i} - c^\alpha(x)u(x) + f^\alpha(x), \tag{1.1.5}$$

$$\mathcal{L}^{\alpha,\beta}u := \sum a_{ij}^{\alpha,\beta}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum b_i^{\alpha,\beta}(x)\frac{\partial u}{\partial x_i} - c^{\alpha,\beta}(x)u(x) + f^{\alpha,\beta}(x), \tag{1.1.6}$$

all the coefficients being uniformly bounded and the linear operators (1.1.5) and (1.1.6) being elliptic. Hamilton-Jacobi-Bellman equations are of the form

$$\sup_\alpha \mathcal{L}^\alpha u = 0, \tag{1.1.7}$$

and Isaacs equations are

$$\sup_\alpha \inf_\beta \mathcal{L}^{\alpha,\beta} u = 0. \tag{1.1.8}$$

Notice that for (1.1.7) the corresponding F is concave in the variables (X, p, r) while for (1.1.8) it is not generally the case.

More concrete examples of geometric origin include

EXAMPLE 1.1.8 (General Monge-Ampère equations). The general Monge-Ampère equation is

$$\det D^2 u = \psi(Du, u, x)$$

where ψ is a given function on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$. This equation is elliptic on the set of convex functions. The most important case is the prescribed Gauss curvature equation

$$\det D^2 u = K(u, x)(1 + |Du|^2)^{\frac{n+2}{2}},$$

the function K being given, and the equation means that K is the Gauss curvature of the graph of u (with respect to an upwards directed normal).

EXAMPLE 1.1.9 (Transport Monge-Ampère equations). Initially the Monge-Ampère equation came into mathematics as a solution of an applied problem of the optimal mass transportation. Let G, G' be domains in $\mathbb{R}^n$ and let $c : G \times G' \to \mathbb{R}$ be a cost function expressing the cost of transportation from a point of G to G'. Let f and g be measures on G and G' such that $\int_G f = \int_{G'} g$. For a measure-preserving transportation function $T : G \to G'$ the cost is given by

$$C(T) = \int_G c(x, T(y)).$$

Problem (Monge, 1784). *Find a transportation function which minimizes the cost.*

Solution. A cost-minimizing function T satisfies $T = \nabla u$ for some function u on G. Moreover, the function u satisfies the so-called *transport Monge-Ampère equation:*

$$\det(D^2 u - c_{xx}(x, T(x))) = \frac{f(x)}{g(T(x))}.$$

As was shown by Brenier [**32**] for the quadratic cost function, the last equation is reduced to

$$\det D^2 u = \frac{f(x)}{g(\nabla u(x))}.$$

EXAMPLE 1.1.10 (Complex Monge-Ampère and Donaldson's equations). The *complex Monge-Ampère equation* has the form

$$\det\left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) = f,$$

where u is a function of complex variables $z_1, \ldots, z_n$ and $f > 0$. When the solution u is a C^2-function, u is a pluri-subharmonic function, i.e., the Hermitian form $\partial_{i\bar{j}} u dz_i d\bar{z}_j$ is positive. The complex Monge-Ampère equation

$$\det\left(g_{i\bar{j}} \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) \det(g_{i\bar{j}})^{-1} = e^{tu+f} \tag{1.1.9}$$

defined on a compact Kähler manifold M with metric $g_{i\bar{j}}$, $t > 0$, plays an important role in the study of the geometry of M; see [**18**], [**276**]. Regularity of solutions of (1.1.9) implies for manifolds with negative first Chern class the existence of a Kähler-Einstein metric.

Donaldson's equation. Let X be a surface. Donaldson, [**75**], considered for a given real parameter ε the following fully nonlinear equation on $X \times [0, 1]$:

$$u_{tt}(1 - \Delta_x u) - |\nabla_x u_t|^2 = \varepsilon, \tag{1.1.10}$$

$x \in X$, $t \in [0, 1]$. The equation (1.1.10) is the equation of geodesics on infinite dimensional space of "Kählerian potentials". It can also be formally considered as a Nahm's equation of motion of a particle on an infinite-dimensional Lie group of area-preserving diffeomorphisms of a surface X.

When X has dimension 1, (1.1.10) is the real Monge-Ampère operator. When X is of dimension 2, (1.1.10) can be reduced to a complex Monge-Ampère operator on $X \times \mathbb{S}^1 \times (0, 1)$.

EXAMPLE 1.1.11 (Hessian equations). The Monge-Ampère equation is a special case of a *Hessian equation*,

$$F(D^2u) := f(\lambda(D^2u)) = \psi(Du, u, x) \tag{1.1.11}$$

where f is a given symmetric function of n variables and

$$\lambda(D^2u) = \lambda = (\lambda_1, \ldots, \lambda_n)$$

denotes the eigenvalues of D^2u. If the function f is continuously differentiable, then the ellipticity of (1.1.11) is equivalent to the condition $f_{\lambda_i} > 0$ for all $i = 1, 2, \ldots, n$. Typical examples of functions f are the elementary symmetric functions

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

and their quotients

$$\sigma_{k,l}(\lambda) := \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}, \quad 1 \le l < k \le n,$$

restricted to the positivity set of the denominator. In the case $k = 1$ we return to (the slightly generalized) Example 1.1.1. The ellipticity of these operators is not obvious and depends on the properties of some functions σ_k. However, one easily checks that all these operators are elliptic on locally uniformly convex functions (all $\lambda_i > 0$).

EXAMPLE 1.1.12 (Equations linear in symmetric functions σ_k). These are Hessian equations of the form

$$\sum_{k=0}^{n} a_k \sigma_k(\lambda) = 0, \qquad a_k \in \mathbb{R},\ k = 0, \ldots, n,$$

which are strictly elliptic for appropriate constants a_k; they appear in some problems of differential geometry. However, they are never uniformly elliptic. From the results of **[208]** one can deduce that these equations can be rewritten in a variational manner. Special Lagrangian equations (see Section 7.1),

$$\operatorname{Im}\{e^{-i\theta}\det(I+iD^2u)\}=0$$

belong to this class. Note that the equation of the form

$$\sigma_k(\lambda)=1$$

sometimes is called the *σ_k-equation.*

EXAMPLE 1.1.13 (Curvature equations). The general form of a curvature equation (or so-called *Weingarten equation*) in Euclidean space is

$$F(u):=f(\varkappa(u))=\psi(Du,u,x)$$

where now $\varkappa(u)=\varkappa=(\varkappa_1,\dots,\varkappa_n)$ denotes the principal curvatures of the graph of u and again f is a given symmetric function of n variables. Since $(\varkappa_1,\dots,\varkappa_n)$ are the eigenvalues of the Hessian D^2u with respect to the metric $I+Du\otimes Du$, this equation is an equation of the form (1.1.1). The *Gauss curvature equation* corresponds to the case $f(\varkappa)=\sigma_n(\varkappa)=\prod_i \varkappa_i$. Other important examples are the *mean curvature*, $\sigma_1(\varkappa)$, yielding a quasilinear elliptic equation, the *scalar curvature* $\sigma_2(\varkappa)$, and the *harmonic curvature* $\sigma_{n,1}(\varkappa)$.

EXAMPLE 1.1.14 (Conformal Hessian equations). Let $n\ge 3, u>0$, and let

$$F[u]:=f(\lambda(A^u))=\psi(u,x),$$

f again being a symmetric function and $\lambda(A^u)=(\lambda_1,\dots,\lambda_n)$ being the eigenvalues of the conformal Hessian

$$A^u:=uD^2u-\frac{1}{2}|Du|^2I.$$

In this case F is invariant under conformal mappings $T:\mathbb{R}^n\longrightarrow\mathbb{R}^n$, i.e., transformations which preserve angles between curves. In contrast to the case $n=2$, for $n\ge 3$ any conformal transformation of $\mathbb{R}^n$ is decomposed into a family of finitely many Möbius transformations, that is, mappings of the form

$$Tx=y+\frac{kA(x-z)}{|x-z|^a},$$

with $x,z\in\mathbb{R}^n, k\in\mathbb{R}, a\in\{0,2\}$, and an orthogonal matrix A. In other words, each T is a composition of a translation, a homothety, a rotation, and (maybe) an inversion.

1.1.2. Uniqueness, existence, and regularity problems. Let us then discuss solutions of nonlinear elliptic equations. There are many problem types for them, but we will study only the most simple (and most fundamental) formulation, namely, the following *Dirichlet problem:*

$$\begin{cases} F(D^2u,Du,u,x)=0 & \text{in } \Omega,\\ u=\varphi & \text{on } \partial\Omega, \end{cases} \tag{1.1.12}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$ and φ is a continuous function on $\partial\Omega$.

A function u is called a *classical* solution of (1.1.12) if $u\in C^2(\Omega)$ and u satisfies (1.1.12). Actually, any classical solution of (1.1.12) is a smooth ($C^{\alpha+2}$) solution, provided that F is a C^α function of its arguments with $\alpha>1, \alpha\notin\mathbb{N}$.

Assuming that $\partial F/\partial u \leq 0$, it is not difficult to prove that (1.1.12) has no more than one classical solution and thus classical solutions verify the fundamental condition of uniqueness. The basic problem is the existence of such classical solutions, and there is no hope of getting such an existence for a sufficiently general class of nonlinear elliptic equations at least for $n \geq 3$ (for $n = 2$ the solutions are classical for uniformly elliptic equations by Nirenberg [**186**]).

The only way out is to define a class of generalized (weak) solutions for the problem (1.1.12) in such a manner that the unicity and existence of solutions may be verified in this class. That is possible for nonlinear elliptic equations (NLEE), which constitutes a major breakthrough in the theory of partial differential equations resulting in the theory of viscosity solutions described in the next section. These solutions are by definition merely continuous functions, and this leads to another major question, the regularity problem, namely, what can be said about differentiability and continuity properties of those generalized solutions. This last problem is very far from a satisfactory answer except for some specific classes of NLEE.

However, there do exist some classes of NLEE with advanced regularity properties, sometimes giving classical solutions. A major part of these results is obtained using the continuity method, a priori bounds, and maximum principles. Now we comment briefly on those fundamental methods.

1.1.3. Continuity method, a priori bounds, and maximum principle(s). The general setting of the *continuity method* is as follows. To prove the existence of a regular solution to a certain elliptic equation, one chooses a continuous family F_t of equations parametrized, say, by a unit interval $t \in [0, 1]$ in such a way that for F_0 regular solutions do exist, F_1 being the initial equation. For example, one can often choose F_0 to be the Laplacian equation $\Delta u = 0$ for which one has very precise and complete information. One then considers the maximal subset $S \subset [0, 1]$ such that for any $t \in S$ the equation F_t has an appropriate solution; therefore $0 \in S$. If one can prove that S is open and closed, then $S = [0, 1]$ by connectedness of the interval and thus the problem is solved. Therefore, to solve the problem one needs to prove

1) S is open in $[0, 1]$,

2) S is closed in $[0, 1]$.

The first point is usually much simpler than the second; often it is a consequence of some general results on implicit functions in appropriate functional spaces. To prove 2) one often uses theorems of Arzelà–Ascoli type; to apply these theorems one needs some upper bounds of appropriate norms of smooth solutions to the initial equation. To be useful, those bounds should not be dependent on the solutions themselves, but on other data such as the ellipticity constant, boundary data, the domain's geometry, etc. Such bounds are called a priori and their proof is usually the only possibility of proving the existence of sufficiently smooth solutions. One finds some examples of a priori bounds in the next section.

Another essential tool of the theory is given by *maximum principles*, the most simple being the maximum principle for harmonic functions, i.e., for Laplace's equation.

THEOREM 1.1.1. *Let u be a solution to the Dirichlet problem* (1.1.12) *for Laplace's equation $\Delta u = 0$. Then u attains its supremum on the boundary of Ω.*

In the nonlinear context one often uses the Alexandrov-Bakelman-Pucci (ABP) maximum principle; see Theorem 1.3.1 below.

1.2. Viscosity solutions

In defining the weak solution to fully nonlinear equations, we have to decide what property of the classical solutions we want to keep instead of considering smooth functions which satisfy the equation. For equations written in variational form one immediately gets an integral identity for the solutions and it is natural to keep such an identity as a characteristic property of the weak solutions. For general fully nonlinear equations that does not work. Fortunately, there is another universal property of the solutions which one can try to use. We will suppose here and below that the functional F depends only on the Hessian D^2u of the unknown function and thus equation (1.1.1) becomes (0.1) of the Preface; the same is supposed for the Dirichlet problem (1.1.12).

Let F be an elliptic operator and let $u \in C^2(\Omega)$ be a *subsolution* ($F(D^2u) \leq 0$) in a domain Ω. Let $v \in C^2(\Omega)$ be a *supersolution* ($F(D^2u) \geq 0$) in Ω. Then $u - v$ attains its supremum on the boundary of Ω. That is the maximum principle for classical solutions of fully nonlinear equations. The idea of viscosity solutions is to extend the notions of sub/supersolutions to a large set of nonsmooth functions preserving the maximum principle.

Notice that weak solutions do not necessarily increase the set of classical solutions. In some situations they are automatically classical. For instance, consider the Laplace operator. The weak solutions in a variational sense are defined as functions $u \in H^1(\Omega)$ which satisfy the following integral identity:

$$\int_\Omega \nabla u \nabla \psi dx = 0,$$

for any smooth function ψ vanishing on $\partial\Omega$. By classical results of Weyl, weak solutions in the sense of this integral identity are smooth functions in any open domain and satisfy the Laplace equation; see, e.g., [**84**]. Therefore, one should verify that the idea of weak solutions works in the general case, increasing sufficiently the set of possible solutions.

One of the classical methods for the solution of the Dirichlet problem for the Laplace equation was suggested by Perron in his 1923 paper [**200**]. He defined a solution to the Dirichlet problem in a bounded domain Ω for the Laplace operator taking the infimum of superharmonic functions which are on the boundary $\partial\Omega$ greater than or equal to the boundary data. Perron proved that there is a unique such infimum which gives a solution to the Dirichlet problem. The *viscosity solutions* to fully nonlinear equations can be defined in a similar way:

DEFINITION 1.2.1. Let G be a bounded domain. A continuous function u in G is a viscosity subsolution of

$$F(D^2u) = f$$

in G if the following condition holds: For any $y \in G$, $\phi \in C^2(G)$ such that $u - \phi$ has a local maximum at y one has

$$F(D^2\phi(y)) \geq f(y).$$