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Volume 7

# The Theory of Multiple Zeta Values with Applications in Combinatorics

Minking Eie

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# Preface

This is a record of my research on multiple zeta values since 2007. In a series of lecture notes used in classes or in seminars on number theory, I provided detailed proofs for both newly found and well-known results. These lecture notes were extremely useful and helpful to my graduate students in learning number theory and writing their thesis. Some particularly interesting sections were published as research papers. These lecture notes are now reworked into a book, which I hope will also be helpful for researchers. This book contains the following important topics concerning multiple zeta values and applications.

1. The duality theorem, the sum formula and the restricted sum formula of multiple zeta values.
2. Shuffle relations produced from shuffle products of multiple zeta values.
3. Double weighted sum formulas of multiple zeta values.
4. Applications of shuffle products in combinatorics.
5. Combinatorial identities of convolution type.
6. Generalizations of Pascal identity.

Around 2005, my former Ph.D student Kwang-Wu Chen and I found that Euler double sums of odd weight can be obtained from integral transforms of products of two Bernoulli polynomials, one with even index and the other with odd index. The evaluations of double Euler sums of odd weight are equivalent to express the corresponding products of Bernoulli polynomials into linear combinations of Bernoulli polynomials of odd indices. Therefore we published several papers concerning evaluations of Euler double sums as well as their analogues. Among other things, a Dirichlet character is added to the first summation of Euler double sums so that it can be evaluated such that either the character is even and the weight is odd, or the character is odd and the weight is even.

Multiple zeta values or $r$-fold Euler sums are natural generalizations of Euler double sums which arose from the knot theory with close relation to Feynman diagrams in quantum physics. In 2007, we began to develop effective ways to explicitly evaluate multiple zeta values of depth $3, 4$ or higher. It ended up with the introduction of multiple zeta values with parameters, something like to replace Riemann zeta functions by Hurwitz zeta functions. We noticed that multiple zeta values of the form $\zeta(\{1\}^m, n+2)$ were much easier to be evaluated. Also certain analogues of $\zeta(\{1\}^m, n+2)$ with parameters can also be evaluated easily. Additional differentiations are needed in order to evaluate multiple zeta values with the sum of depth and weight is odd.

Due to Kontsevich, multiple zeta values are expressed as iterated integrals over simplices of weight dimensions of strings of differential forms. Only two kinds of differential forms appear: $dt/(1-t)$ or $dt/t$. Moreover it always begins with $dt/(1-t)$ and ends up with $dt/t$. Once multiple zeta values are expressed as iterated integrals over simplices, the shuffle product of two multiple zeta values is equivalent to find all possible interlacings of two sets of variables. So two multiple zeta values of weight $m$ and $n$ will produce $\binom{m+n}{m}$ multiple zeta values of weight $m+n$ after their shuffle product. Based on such a simple fact, we are able to produce combinatorial identities from shuffle relations obtained from shuffle products of certain multiple zeta values.

Some particular multiple zeta values such as $\zeta(\{1\}^m, n+2)$ or sums of multiple zeta values can be expressed as integrals in one or two variables so that their shuffle products can be carried out more efficiently. By counting numbers of multiple zeta values produced from shuffle products, we may obtain combinatorial identities with a single binomial coefficient on one side and sums of products of binomial coefficients on the other side.

The theory of multiple zeta values is not fully developed. Devoted wholly to this subject, this book aims to introduce a systematic theory of multiple zeta values as well as applications of shuffle products to combinatorics. Hopefully, this will lead to further developments of the theory and provide inspiration to both experts and amateurs.

Finally, I would like to thank my graduate students who helped to type the whole book, thank my Ph.D student Chung for compiling the files and thank my colleague Professor Chang for the final editing.

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October 29, 2012
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# Contents

# Part I

# Basic Theory of Multiple Zeta Values

The theory of multiple zeta values began with the evaluation of Euler double sums in terms of special values at positive integers of Riemann zeta function, proposed by Goldbach in 1742 to Euler. Until 1992, multi-versions of Euler double sums appeared and are called multiple zeta values or $r$-fold Euler sums.

A multiple zeta value of depth $r$ and weight $w$ can be expressed as a sum of multiple zeta values of lower depth and the same weight when $r + w$ is odd. Some particular multiple zeta values and sums of multiple zeta values can be expressed in terms of single zeta values, the special values at positive integers of Riemann zeta function. All these assertions need further investigation through integral representation of multiple zeta values due to Kontsevich around 1996.

The shuffle product of two multiple zeta values enables us to express the product of two multiple zeta values of weight $m$ and $n$ as a sum of $\binom{m+n}{m}$ zeta values of weight $m+n$. How to carry out the shuffle product becomes an intricate problem. Fortunately, we are able to develop an alternative to overcome such difficulty.