# DYNAMICS OF MARINE VEHICLES

RAMESWAR BHATTACHARYYA

A VOLUME IN
OCEAN ENGINEERING:
A WILEY SERIES EDITED BY
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# DYNAMICS OF MARINE VEHICLES

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A WILEY-INTERSCIENCE PUBLICATION

**JOHN WILEY & SONS, New York · Chichester · Brisbane · Toronto**

**Library of Congress Cataloging in Publication Data**

Bhattacharyya, Rameswar.
Dynamics of marine vehicles.

(Ocean engineering, a Wiley series)
"A Wiley-Interscience publication."
Includes bibliographical references and index.
1. Ships—Seakeeping. I. Title.

VM156.B49 623.8′1 78-950
ISBN 0-471-07206-0

Photosetting by Thomson Press (India) Limited, New Delhi
Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

# SERIES PREFACE

Ocean engineering is both old and new. It is old in that man has concerned himself with specific problems in the ocean for thousands of years. Ship building, prevention of beach erosion, and construction of offshore structures are just a few of the specialties that have been developed by engineers over the ages. Until recently, however, these efforts tended to be restricted to specific areas. Within the past decade an attempt has been made to coordinate the activities of all technologists in ocean work, calling the entire field "ocean engineering." Here we have its newness.

Ocean Engineering: A Wiley Series has been created to introduce engineers and scientists to the various areas of ocean engineering. Books in this series are so written as to enable engineers and scientists easily to learn the fundamental principles and techniques of a specialty other than their own. The books can also serve as text books in advanced undergraduate and introductory graduate courses. The topics to be covered in this series include ocean engineering wave mechanics, marine corrosion, coastal engineering, dynamics of marine vehicles, offshore structures, and geotechnical or seafloor engineering. We think that this series fills a great need in the literature of ocean technology.

MICHAEL E. MCCORMICK, EDITOR
RAMESWAR BHATTACHARYYA, ASSOCIATE EDITOR
*November 1972*

# PREFACE

This book is based on lecture notes prepared for students of naval architecture, marine engineering, and ocean engineering at the University of Michigan, the U.S. Naval Academy, the Catholic University of America, the University of Veracruz, Mexico, and the Technical University at Guayaquil in Ecuador, over a period of 10 years. It must be emphasized that the literature presently available on the dynamics of marine vehicles is either more scientifically oriented or scattered in various publications not easily accessible to the common reader.

Primarily the book is intended as a textbook for a first course in seakeeping, and emphasis is therefore placed on the fundamentals of the subject matter. Through the use of numerous figures, tables, and solutions of exemplary problems, the aim is to help the reader in understanding the basic principles and to demonstrate the applicability of the methods outlined in the text, since I am strongly of the opinion that any treatise on an engineering discipline should consider its primary goal to be oriented toward numerical results. However, after the fundamentals have been grasped in this way, the reader can go on to study the advanced developments made in recent years for the purpose of obtaining more scientific, if not more accurate, solutions to actual problems.

The book is self contained; all essential material related to the topics covered has been defined and derived in the text. Also, the various chapters have been arranged in such a way that the reader becomes acquainted with the various topics step by step, and I followed strictly a course that would qualify the book to be self-taught with a basic knowledge of calculus. The chapters have been arranged to present a sequence of the physical phenomena necessary for a complete understanding of dynamics of marine vehicles (e.g., simple waves, linearized motion, nonlinear motion, coupled motion, the seaway, motions in an irregular seaway), and each chapter can be well understood with the knowledge acquired from studying the preceding chapters.

Along with the sophisticated marine systems, the design procedures for marine vehicles should not be limited to stability, structure, resistance, and propulsion; an integrated design criterion should also include a study of motion and maneuvering. It is not only desirable but also feasible to make accurate quantitative predictions of the seakeeping and maneuverability qualities of a marine vehicle when geometrical descriptions of the hull form, the weight distribution, and the necessary seaway conditions are available. To do this requires a good understanding of the fundamentals of the subject, the theories making it possible to justify all the assumptions necessary for quantitative answers, and a proper skill and ingenuity in the complex design process. This book has been written also with the idea that a design engineer will have ample opportunity to understand the basic principles and to follow the proper steps to improve the hull form for his specific design.

I should like to express my indebtedness to all the authors of the various publications listed in the bibliography. They have provided me with the necessary knowledge to write this book. Specifically I express my sincere gratitude to Rear Admiral R. W. King, USN (Ret.), and Professor M. E. McCormick of the U.S. Naval Academy, without whose encouragement and initiative this book could not have been written. Acknowledgments are due also to the students who have taken my course at the University of Michigan, the U.S. Naval Academy, the Catholic University of America, and the University of Veracruz in Mexico; especially Mr. D. Goldstein of the Naval Ship Engineering Center, Mr. F. Agdern of the Naval Facility Engineering Command, Ensign M. C. Tracy, and Señor R. Hernandez Valdes of the University of

Veracruz helped me throughout in the preparation of the manuscript.

I am grateful to Mrs. I. E. Johnson for superb typing and the necessary secretarial help from the very beginning. Particular thanks go also to Mrs. D. V. Christensen and Mrs. V. E. Stafford for their expert editorial assistance.

Any suggestions or corrections from readers in regard to any part of this book will be highly appreciated.

RAMESWAR BHATTACHARYYA

*Annapolis, Maryland*
*January 1978*

*Acknowledgment is due to the following Organizations:*

Royal Institution of Naval Architects

Figures 11–19, 13–11, 14–7, 14–18, 14–20b, 14–21, 14–23, 16–6, 16–12, 16–15, 16–16, 16–21, 16–29, 16–30, 16–41, 16–43, 16–44, 16–61, 16–62, 16–65, 16–67, 16–68, 17–10b
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Institution of Marine Engineers

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Association Technique Maritime at Aéronautique

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Society of Naval Architects of Japan

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North East Coast Institution of Engineers and Shipbuilders

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Schiffstechnik

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Schiffbautechnische Gesellschaft

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Royal Institution of Engineers (The Netherlands)

Figures 11–8, 16–18

Hamburgische Schiffbauversuchsanstalt

Figures 4–4, 4–6

Handbuch der Werften

Figures 4–5, 4–8, 4–21, 4–22

International Shipbuilding Progress

Figures 4–26, 4–28, 7–24, 9–6 to 9–8, 9–11, 13–2, 13–5, 13–6, 16–2 to 16–5, 16–7, 16–19, 16–20, 16–31, 16–32, 16–35, 16–36, 16–38

Society of Naval Architects and Marine Engineers

Figures 4–10, 4–23, 4–24, 4–31, 4–35, 4–39, 4–40, 6–4, 6–8, 6–9, 7–12, 7–19, 7–29, 8–3 to 8–6, 8–9 to 8–11, 9–5, 11–3a, 11–6, 13–7 to 13–10, 14–3a, 14–13, 16–2, 16–3, 16–5, 16–10, 16–13, 16–14, 16–17, 16–33, 16–34, 16–39, 16–40, 16–46, 16–48, 16–49, 16–51, 16–55 to 16–58, 16–63, 17–3 to 17–7, 17–15

Davidson Laboratory

Figure 17–14

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# Chapter One

# INTRODUCTION

Ships are built for the purpose of carrying men, material, and/or weapons upon the sea. In order to accomplish its mission, a ship must possess several basic characteristics. It must float in a stable upright position, move with sufficient speed, be able to maneuver at sea and in restricted waters, and be strong enough to withstand the rigors of heavy weather and wave impact. To design a ship with these features, the naval architect must have an understanding of ship dynamics.

With a simple knowledge of hydrostatics a naval architect can produce a ship that will float upright in calm waters. However, ships rarely sail in calm water. Waves, which are the main source of ship motions in a seaway, affect the performance of a ship considerably and the success of a ship design depends ultimately on its performance in a seaway. Unfortunately, however, the prediction of ship motions, resistance and power, and structural loads in an actual seaway is such a complex problem that the naval architect is usually forced to select the hull form and ship dimensions on the basis of calm water performance without much consideration of the sea and weather conditions prevailing over the route on which the ship is to operate.

To study the effects of waves on ship dynamics it is logical that we should also understand ocean waves, which are not regular but highly complex in nature. Statistical means have been adopted to study this irregular behavior of the seaway and also to obtain ship motion characteristics.

It is not the motion characteristics per se that are important in the study of ship behavior in a seaway but rather the dynamic effects caused by the motions themselves. These effects are the shipping of green water on the deck, the emergence of the forefoot leading to slamming, and the effects of acceleration due to pitch, heave, or roll, or all combined.

When the relative movement of the bow and local wave surface become too great, water is shipped over the forecastle. The shipping of green water can have a very detrimental effect if watertight integrity is not maintained. Many of the electrical systems can be so damaged that they may not be functional. The freezing of green water on contact can stop a piece of apparatus on the weather decks from functioning and may seriously impair the fighting quality of a warship. Also, in an earlier stage, spray is driven over the forward portion of the ship by the wind. Both conditions (spray and green water) are undesirable and can be improved by increasing the freeboard.

Under some conditions, the pressures exerted by the water on a ship's hull may become excessive and slamming may take place. Slamming is characterized by a sudden change in the vertical acceleration of the ship, followed by a vibration of the hull girder in its natural frequency. The conditions leading to slamming are high relative velocity between the ship and the water surface, shallow draft, and small deadrise.

Repeated slams will not only damage the ship structure and other components, but also will have a considerable effect on the personnel operating the ship. This is especially important in respect to the satisfactory operation of naval vessels, the mission of which is to act as a floating platform for weapon systems. The platform, therefore, should be as stable as possible. The area between 10 and 25% of the length from the bow is the one most likely to suffer high pressures leading to damage.

In the design of ships, speed is an important factor. However, there is a loss of speed while a vessel is under way in a sea, because of the increase in motion resistance and the loss of propeller efficiency. This results in higher fuel consumption and thereby limits the cruising range. The heavier the seaway, the greater is the loss of speed. To overcome this loss it is often necessary to improve the resistance and propulsion characteristics of the vessel, as well as to design the machinery plant for adequate reserve power. Although model tests can predict with reasonable accuracy the still water resistance and propulsion performance of a ship, their determination in a seaway is still the subject of research. The maximum speed that can be attained by a ship is governed, not necessarily by the available power, but mostly by the accelerations experienced in a seaway.

To reduce the dynamic effects, various means of motion stabilization have been adopted. Bilge keels, damping tanks, and fins are a few examples. Knowledge of the phenomenon of resonance between regular waves and the rolling motion has led to the use of successful roll damping devices, but not much progress toward dampening the pitching motion adequately has yet been made because of the large forces involved. Consideration of motion stabilization is particularly important for passenger ships, as well as many types of naval vessels. Structural failure in severe seas is not infrequent even with modern technology.

The relative importance of the various aspects of ship performance in a seaway varies from design to design, depending on what the operators require of the ship. The following general items, must be investigated when designing seaworthy ships:

a. Excessive motions, which are undesirable since they may impair stability and cause discomfort to the crew and passengers. Also, in warships most weapon systems require a stable platform for proper functioning.

b. Additional stresses caused by the ship's bending or by wave impacts in a seaway.

c. Inertial forces causing damage to equipment, armament structures, and so forth.

d. Shipping and spraying of green water, causing equipment breakdown and degradation of liability.

e. Slamming.

f. Speed reduction and the conditions under which the propeller will start racing, thereby overloading the propelling machinery and hence increasing the fuel consumption per mile or dropping off the cruising range.

g. Ship-handling quality.

The various problems encountered in regard to ship motions may be investigated in four different ways:

1. Analytically, that is, on a theoretical basis.

2. Experimentally, by means of model tests in controlled environments.

3. Empirically, through statistical observations.

4. Directly, as with trials of ships after they are built.

Both theoretical and experimental studies help the designers to determine the influences of various ship features on seakeeping characteristics, knowledge that is extremely valuable in designing a ship. Therefore one of the most important studies in naval architecture is the investigation of ship performance in rough water. Both merchant and naval vessels must maintain a high degree of seakeeping quality in many different types of weather and still attain their mission—the merchant ship from the commercial point of view, and the naval vessel with regard to optimum operational ability. For the purpose of design one should be able to estimate the dynamic forces to which a ship may be subjected and the motions that result therefrom. Theoretical studies, model results, and full-scale data are all necessary to provide reliable design criteria.

In recent years, research on ship motions has made considerable advances in the area of theoretical development, as well as in experimental facilities. However, no quantitative index has yet been found to compare the seakeeping qualities of ships, as is possible in comparing the resistance or propulsion characteristics of one hull form with another by means of simple coefficients.

The introduction of advanced marine vehicles, such as planing crafts, hydrofoil boats, and air cushion vehicles has necessitated further studies in seakeeping in order to achieve the maximum results from these special vehicles. Intensive investigations are now under way to determine experimentally the effects of parametric variations in motions, bottom pressures, and power requirements on models of planing boats, surface effect ships (SESs), and so on. In addition, scale effect studies on high-performance vessels are being looked into to correlate test results from models of different scales and full-scale trials.

Therefore, the responsibility of a ship designer includes the development of technology for measuring, predicting, and improving the various qualities that govern ship dynamics. This also includes the application of this technology to specific designs, the identification of design faults, and the correction and improvement of such designs. The specifics will depend on the particular design, but it is essential that the designer have some means of judging the expected performance.

In theoretical investigations the problem of determining the motions of a ship consists of deriving simple analytical expressions for the surface of the seaway and determining the ship motions for such a seaway. Theoretical studies can offer the following:

a. General information regarding the most relevant characteristics of the behavior of a ship in a seaway.

b. A prediction of the motion of a ship in any given seaway.

c. An insight into the acceptable values of motions, accelerations, and so on.

d. A knowledge of the average performance to be expected, including stability and resistance.

e. Basic ideas regarding motion stabilization and ways to achieve it.

f. Guidelines for model tests and full-scale trials.

However, since ship motion is rather complex, it cannot be completely treated by analytical means alone; therefore model experimentation and ship trials are carried out in order to predict ship performance. Sophisticated methods of model tests have been developed in various experimental facilities throughout the world and extensive ship trials are conducted in order to correlate model and ship results. This is especially important for naval ships. However, before new ships, including hydrofoils, SESs, and hovercrafts, can be employed effectively in their design environment (open sea and at speed), the human habitability factor must be addressed.

It is to a basic and fundamental discussion of such means that the later chapters are devoted.

Chapter Two

# SIMPLE HARMONIC MOTION

## 2.1 INTRODUCTION

Motions of a body can be described as either translational or rotational. These motions, according to Newton's law, take place continuously in one direction only unless disturbed by some external force. The direction of motion can also be alternating; that is, motion can progress in one direction and then reverse after an interval of time. Such a motion is known as *oscillatory*. An oscillatory motion is common in nature, and, since it was originally studied in relation to music, is also called a *harmonic motion*. Since most harmonic motions are rather complicated, a simplified treatment is adopted here using some simple oscillatory motions, which are then called *simple harmonic motions*.

In the case of simple harmonic motion, when a body is displaced from its equilibrium position, a force that is inherent in the body tends to bring it back to its original equilibrium position. This force, known as the *restoring force*, is directly proportional to the displacement of the body from its equilibrium position. When displaced from its equilibrium position, the body moves back toward this position with an acceleration under the action of the restoring force, that is, the body moves faster and faster as it nears its equilibrium position. However, as the body comes closer to this position, the restoring force decreases and the acceleration toward the equilibrium position diminishes. When the body finally reaches its equilibrium position, the restoring force and acceleration vanish, but by then the body has attained its maximum velocity. If there is no force to stop the body at its equilibrium position, it will move past this position in the opposite direction. Again a restoring force acting toward the equilibrium position comes into play, but in the opposite direction. As the body continues to move further, the displacement and the restoring force, as well as the acceleration, increase until the velocity becomes zero, that is, the body reaches its maximum displacement from its equilibrium position. Now, under the influence of the restoring force, the body gains velocity continuously until it reaches the equilibrium position and moves past this position again.

If there is no resistance or damping during this oscillatory motion, the body will oscillate indefinitely, and the maximum displacement of the body on either side of its equilibrium position will remain constant. The time taken to reach from one extreme point to the one on the other side and back is known as the *period* of the motion.

In a simple harmonic motion, displacement, velocity, and acceleration change constantly at every instant. This kind of motion can be represented by another kind of motion with a constant speed: motion around a circle.

This topic is discussed in the following section.

## 2.2 EQUATIONS OF SIMPLE HARMONIC MOTION

If a point is considered to be moving along the circumference of a circle with uniform speed, the motion of the projection of the point on the diameter of the circle is defined as the simple harmonic motion.

In Fig. 2.1 let us suppose that point $P$ is moving along the circumference of a circle having a radius $z_a$. If $P$ has a constant angular velocity of $\omega$ radians

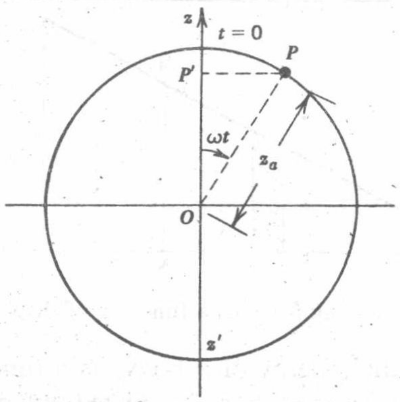


**Figure 2.1** Definition of simple harmonic motion.

per unit time, $P'$ is the projection of $P$ on the diameter of the circle $zz'$. While $P$ rotates along the circumference of the circle, $P'$ moves from $z$ to $z'$ and again back to $z$. The motion of $P'$ is known as simple harmonic motion. Now

$$\text{Displacement of } P' \text{ from } O = OP' = z = z_a \cos \omega t \quad (2.1)$$

$$\text{Velocity of } P' = \frac{dz}{dt} = -z_a\omega \sin \omega t \quad (2.2)$$

$$\text{Acceleration of } P' = \frac{d^2z}{dt^2} = -z_a\omega^2 \cos \omega t \quad (2.3)$$

Figure 2.2 shows displacement, velocity, and acceleration. The amplitude of $P'$, its maximum displacement from the middle position, is $z_a$. The period of motion, the time required for $P'$ to reach $z'$ from $z$ and move back to $z$ again, is the same as the time required for $P$ to make a complete rotation:

$$T = \frac{2\pi z_a}{\omega z_a} = \frac{2\pi}{\omega}$$

$$\text{Characteristic frequency} = \frac{1}{T}$$

$$\text{Angular frequency} = \omega = \frac{2\pi}{T}$$

Note the following:

a. The maximum velocity occurs when $v = z_a\omega \sin \omega t$ is maximum, that is, when $\sin \omega t = \pm 1$. This occurs when $P'$ is at the midpoint from its two extreme positions, or when the body is at its equilibrium position.

b. The maximum acceleration occurs when the expression $\cos \omega t = \pm 1$, that is, when the body is at its extreme position from the position of equilibrium.

c. As already mentioned, the restoring force in a simple harmonic motion is directly proportional to the displacement of the body from its position of equilibrium, that is,

$$f = cz$$

where $f$ is the restoring force, $c$ is a constant, and $z$ is the displacement of the body from $O$, which is the position of equilibrium. When the body is at its equilibrium position, $z$ is zero and so is the restoring force $f$. When the body is at its extreme position (i.e., $z = z_a$), the restoring force is maximum and equal to $cz_a$.

Therefore, as the body moves from its position of equilibrium to one extremity, it acts against a force that is zero at first and then increases gradually to a maximum value of $cz_a$. Then from (2.3) we see that the maximum acceleration is $-\omega^2 z$. It is noted that the amplitude $z_a$ does not come into this equation, or into the equation for period $T = 2\pi/\omega$. Therefore one can say that, since $\omega^2$ does not refer to any particular circle, a whole set of simple harmonic motions, even ones of different amplitudes, will have the same period.

Hence the simple harmonic can be redefined as follows:

"Simple harmonic motion is motion in a straight line if at each instant the acceleration is directly proportional to its distance from a fixed reference point in the straight line and acts toward that point."

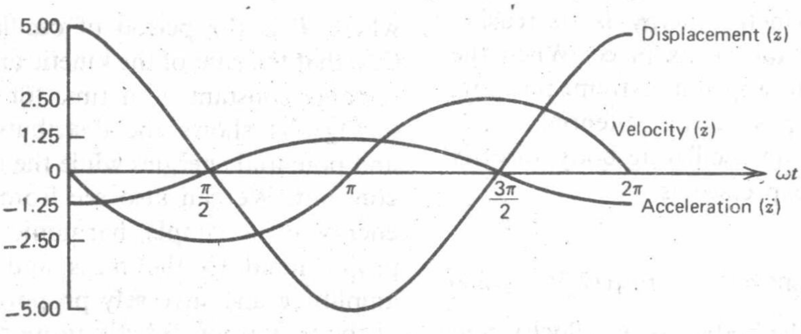


**Figure 2.2** Displacement, velocity, and acceleration of a simple harmonic motion.

Also,

"Whatever be the amplitude of motion, the period is determined only by the acceleration at unit displacement [i.e., putting $z = 1$ in (2.3)]."

The preceding discussion has dealt with simple harmonic motion in a straight line, that is, oscillatory translational motion along the diameter $zz'$ of the circle in Fig. 2.1, where the body oscillates in a straight line, but oscillatory rotational motion, that is, the motion of a body moving back and forth along a circular arc, can also be simple harmonic motion. In that case the restoring force is the restoring moment, the angular acceleration is proportional to the angular displacement and oppositely directed, the angular motion is simple harmonic, and again the period is independent of the angular amplitude.

**Example 2.1**

A simple harmonic motion $z = z_a \cos \omega t$ has an amplitude of 5 ft and a circular frequency of 0.5 rad/sec. Show with the help of a diagram how the displacement, velocity, and acceleration of this simple harmonic motion should vary with time.

*Solution*

Angular velocity: $\omega = 0.5$ rad/sec (given)

Amplitude of motion: $z_a = 5$ ft (given)

Displacement: $z = z_a \cos \omega t = 5 \cos 0.5t$

Velocity: $\dfrac{dz}{dt} = -\omega z_a \sin \omega t = -2.5 \sin 0.5t$

Acceleration: $\dfrac{d^2z}{dt^2} = -\omega^2 z_a \cos \omega t = -1.25 \cos 0.5t$

See Fig. 2.2.

When the body begins to move away from its equilibrium position, it possesses only kinetic energy. Since it loses velocity as it goes further from the neutral position, the kinetic energy is decreasing and potential energy is taking its place. When the body comes to a complete stop at its extreme position, all the energy the body possesses is potential.

The kinetic energy of an oscillating body, directly as a function of time, is expressed as

$$E_K = \tfrac{1}{2}mv^2$$
$$= \tfrac{1}{2}m(\omega^2 z_a^2 \sin^2 \omega t) \qquad \text{from (2.2)} \qquad (2.4)$$

where $m$ is the mass of the body, $v$ is its velocity, and $z_a$ is its amplitude or maximum displacement.

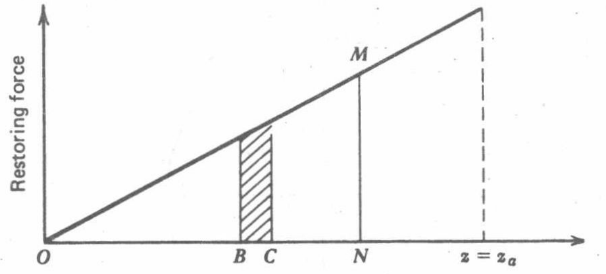


**Figure 2.3** Restoring force as a function of displacement.

The potential energy of a body, as a function of time (or displacement), is calculated by equating the work done against the restoring force $F$, expressed as

$$F = -ma$$
$$= -m(-\omega^2 z_a \cos \omega t) \qquad \text{from (2.3)}$$
$$= m\omega^2 z \qquad (2.5)$$

As stated before, the restoring force is directly proportional to the displacement of the body from the reference point or the equilibrium position as shown in Fig. 2.3. In this figure the shaded area represents the work done or the increase of potential energy corresponding to the small change in displacement from $B$ to $C$. With the same argument the increase in potential energy of a body from its equilibrium position to any displacement $z$ (i.e., $ON = z$) is given by the area of the triangle $OMN$ or

$$E_p = \tfrac{1}{2}zF$$
$$= \tfrac{1}{2}zm\omega^2 z$$

Since $z = z_a \cos \omega t$,

$$E_p = \tfrac{1}{2}m\omega^2 z^2$$
$$= \tfrac{1}{2}m\omega^2 z_a^2 \cos^2 \omega t \qquad (2.6)$$

Adding (2.4) and (2.6), we obtain the total energy:

$$E_K + E_p = \tfrac{1}{2}m\omega^2 z_a^2$$

or

$$= \frac{2\pi^2 m z_a^2}{T^2} \qquad (2.7)$$

where $T$ is the period of oscillation. We see from (2.7) that the sum of the kinetic and potential energies remains constant with time (or displacement); that is, Fig. 2.4 shows the distribution of both kinetic and potential energies while the total energy remains constant. We can also see from (2.7) that the total energy of a simple harmonic motion is directly proportional to the mass and the square of the amplitude and inversely proportional to the square of the period (or directly proportional to the square of the frequency).

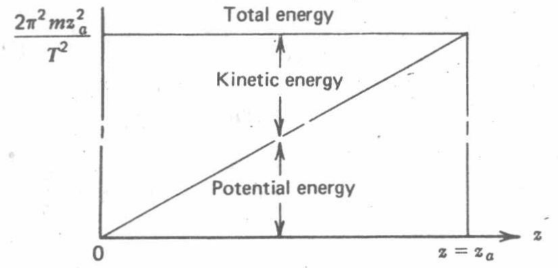


**Figure 2.4** Distribution of kinetic and potential energies.

### Phase Difference

We saw in Fig. 2.1 that, if point $P$ starts rotating in the clockwise direction from position $z$, the time $t$ when $P$ is at $z$ equals zero and the displacement of $OP'$ for the simple harmonic motion is given as

$$OP' = z = OP \cos \omega t$$
$$= z_a \cos \omega t$$

However, if $P$ starts its motion from $Q$, as in Fig. 2.5, then the time $t$ when $P$ is at $Q$ equals zero and

$$OP' = z = OP \cos(\omega t + \varepsilon)$$
$$= z_a \cos(\omega t + \varepsilon)$$

Figure 2.6 shows the two curves drawn on the same axes for comparison, where

$$z_a = 2, \qquad \varepsilon = \frac{\pi}{3}, \qquad \theta = \omega t$$

We notice that the two curves have the same shape and size but are moved or shifted in relation to each other in the direction of the $\theta$-axis. Both functions have the same amplitude and the same period in $\theta$, but they are different in phase. In any general case we say that, if $z = z_a \cos(\theta + \varepsilon)$, the constant angle $\varepsilon$ is called the *phase angle* of the function with respect to $z = z_a \cos \theta$. Generally, the range of $\varepsilon$ is restricted, so that $-\pi \leqslant \varepsilon \leqslant \pi$. The phase shift is $-\varepsilon$, which is the condition for the argument of the cosine function to be zero.

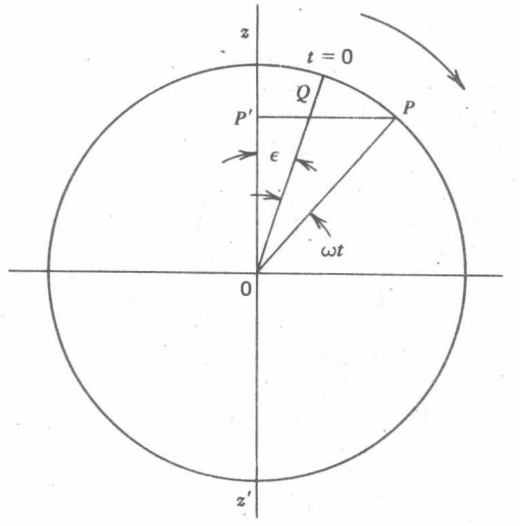


**Figure 2.5** Phase difference in a simple harmonic motion.

*Note*: If we compare $z_a \cos 3\theta$ and $z_a \cos(3\theta + \pi/4)$, the phase angle is $\pi/4$. However, this is not the shift of the graph to the left from the graph of $z_a \cos 3\theta$. The left shift is given by $3\theta + \pi/4 = 0$ or $\theta = -(\pi/12)$. If $\varepsilon$ is positive (as in Fig. 2.6, $\varepsilon = \pi/3$), the function $z_a \cos(\theta + \varepsilon)$ is said to lead the function $z_a \cos \theta$, where the corresponding zeros of $z_a \cos(\theta + \varepsilon)$ occur *before* or to the *left* of the zeros of $z_a \cos \theta$. If $\varepsilon$ is negative, $z_a \cos(\theta + \varepsilon)$ *lags* the function $z_a \cos \theta$.

With the above explanation, it is said that $z_a \sin \theta = z_a \cos(\theta - \pi/2)$ lags $z_a \cos \theta$ by $\pi/2$, and $z_a \cos \theta = z_a \sin(\theta + \pi/2)$ leads $z_a \sin \theta$ by $\pi/2$. Note also that a lead by $\pi$ has the same effect as a lag by $\pi$. The following conclusions can be drawn from Fig. 2.6:

a. A change in the amplitude leaves the period in $\theta$ and the phase angle unchanged.

b. A change in the period in $\theta$ leaves the amplitude and the phase angle unchanged.

c. A change in the phase angle leaves the amplitude and the period in $\theta$ unchanged.

The amplitude, the period in $\theta$, and the phase angle are independent properties of the functions,

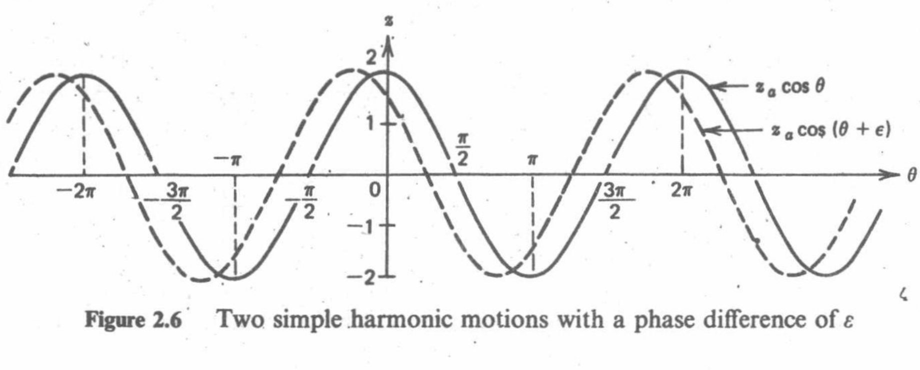


**Figure 2.6** Two simple harmonic motions with a phase difference of $\varepsilon$

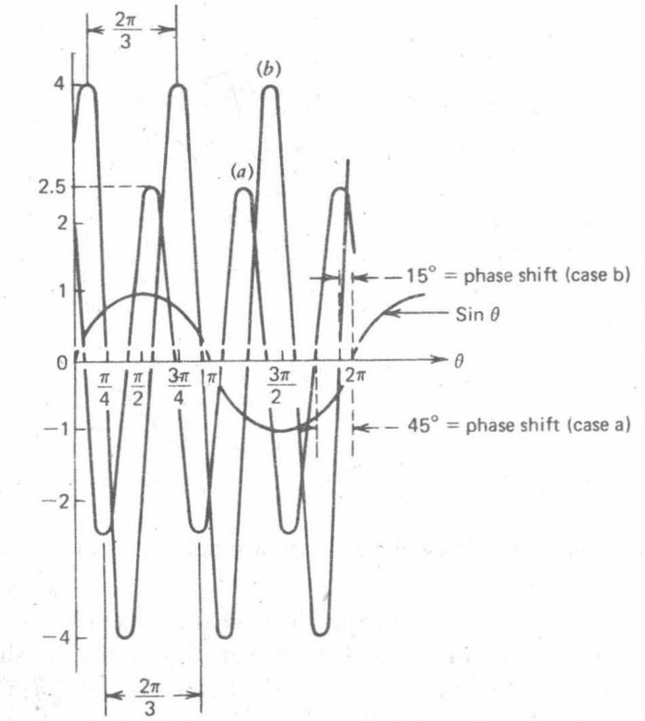


**Figure 2.7** Phase shift.

whereas the phase shift in $\theta$ is dependent on both the phase angle and the period in $\theta$ but is independent of the amplitude.

**Example 2.2**

Find the amplitude, the period, the phase angle, and the phase shift relative to $\sin\theta$ of:

a. $2.5 \sin 3(\theta + \pi/4)$.
b. $4 \sin(3\theta + 45°)$.

*Solution* (see also Fig. 2.7.):

a. $2.5 \sin 3\left(\theta + \frac{\pi}{4}\right)$ Amplitude $= 2.5$

Phase Shift: $3\theta + 3\frac{\pi}{4} = 0$

or $\theta = -\frac{\pi}{4}$

$\text{Period} = \frac{2\pi}{3}$

Phase angle: $3\theta + \frac{3\pi}{4}$

Therefore, phase angle $= \frac{3\pi}{4}$

b. $4 \sin(3\theta + 45°)$ Amplitude $= 4$

Phase shift: $3\theta + 45° = 0$

or $\theta = -\frac{45°}{3}$

$= -15°$

$\text{Period} = \frac{2\pi}{3}$

Phase angle $= 45°$

In Fig. 2.8 the harmonic curves are shown as each having the same amplitude and period but a different origin of time. In Fig. 2.8*a* time is measured when point $P'$ is in a position $z = z_a$. In Fig. 2.8*c* time is measured when $P'$ is in the equilibrium position (i.e., $z = 0$); here, as in Fig. 2.8*b*, time is measured when $t_0$ units of time have elapsed after $P'$ has passed the extreme position $z = z_a$ in the clockwise direction, or

$$\varepsilon = \omega t_0$$

$$t_0 = \frac{\varepsilon}{\omega} = \frac{\varepsilon T}{2\pi}$$

As can be seen from Fig. 2.8, all three diagrams describe identical curves except that they are shifted in relation to each other along the abscissa, that is, the $t$-axis. As said before, this relative shift is known as the phase shift, and when this is multiplied by the angular velocity $\omega$ (which is the same as the circular frequency), the phase angle $\varepsilon$, in terms of radians, is obtained. Note again that the period $T$ is the same

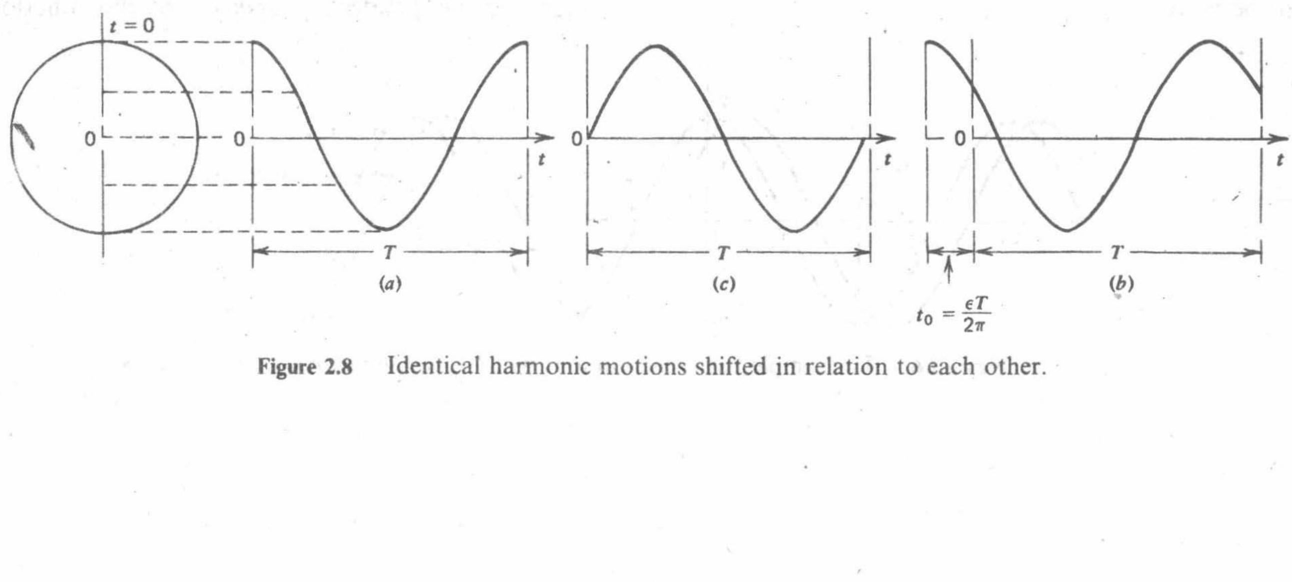


**Figure 2.8** Identical harmonic motions shifted in relation to each other.

in all three diagrams in Fig. 2.8. If this is not the case, the curves cannot be compared to each other. Therefore the phase angle is applicable only if harmonic motions of the same frequency are considered. Thus, if we compare (2.1) and (2.2), we have

$$z = z_a \cos \omega t$$

$$\dot{z} = -\omega z_a \sin \omega t$$

$$= \omega z_a \cos\left(\omega t + \frac{\pi}{2}\right)$$

This means that the velocity of $P'$ is a harmonic function with the same frequency as displacement. As shown in Fig. 2.2, the amplitude of the acceleration is $\omega^2$ times larger than that of the displacement and the phase angle is 180°, whereas the frequency in all three cases remains the same.

**Example 2.3**

The simple harmonic motion of a body is expressed as

$$z = z_a \cos \omega t$$

and the period $T = 2$ sec. If the body is held fixed for a time $t = 0.5$ sec and then is released, and if the time is measured from the same starting point as before, find the expression for the continuing harmonic motion.

*Solution:*

As explained, the phase angle $\varepsilon$ is described as

$$\varepsilon = \omega t_0$$

$$= \frac{2\pi}{T} t_0 = \frac{2\pi}{2} \times 0.5 = \frac{\pi}{2}$$

Therefore the simple harmonic motion will be

$$z = z_a \cos\left(\omega t + \frac{\pi}{2}\right)$$

## 2.3 VECTOR REPRESENTATION

As mentioned earlier, the simple harmonic motion of any oscillating body can be represented by the projection of the end point of a radial vector on the diameter of the circle (see Fig. 2.9). Any one of the projections of $P$ will represent the harmonic motion. If, for example, $t = 0$ when $P$ is at $z$ and $P$ rotates in the clockwise direction, the vertical projection will be $z_a \cos \omega t$ and the horizontal one will be $z_a \sin \omega t$. The latter may be represented by the vertical

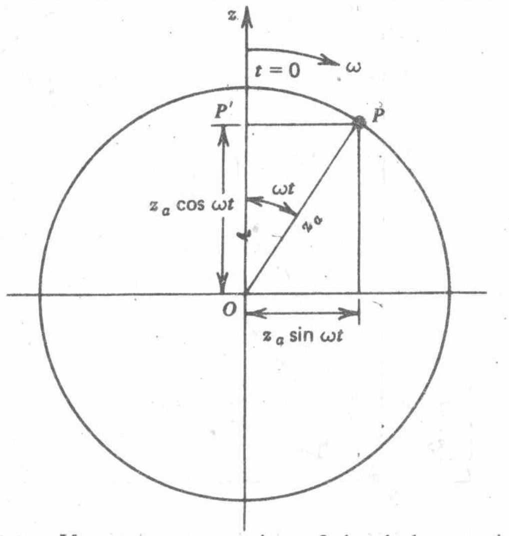


**Figure 2.9** Vector representation of simple harmonic motion.

vector with a phase angle of $-(\pi/2)$ since

$$z_a \sin \omega t = z_a \cos\left(\omega t - \frac{\pi}{2}\right)$$

For example, from (2.1), (2.2), and (2.3) the rotating vectors for displacement, velocity, and acceleration may be described as in Fig. 2.10.

If we differentiate the displacement vector $z_a$, we see that the velocity vector length is multiplied by $\omega$ and advances by an angle of 90°. The vector representation is useful if we are to add and subtract the simple harmonic motions with phase differences. The resultant vector will represent the resultant motion both in amplitude and in phase.

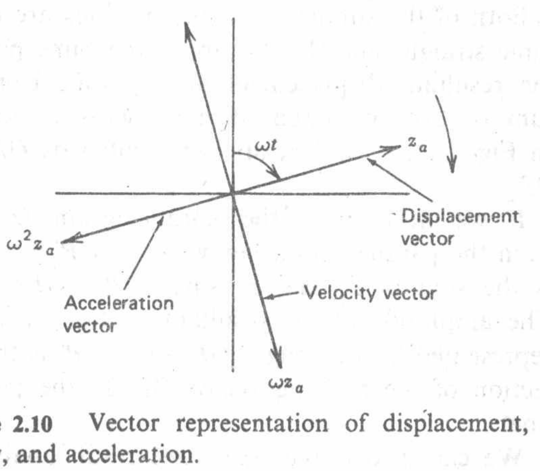


**Figure 2.10** Vector representation of displacement, velocity, and acceleration.

## 2.4 ADDITION OF SIMPLE HARMONIC MOTIONS

When two simple harmonic motions have the same period, they can differ only in amplitude and in phase.

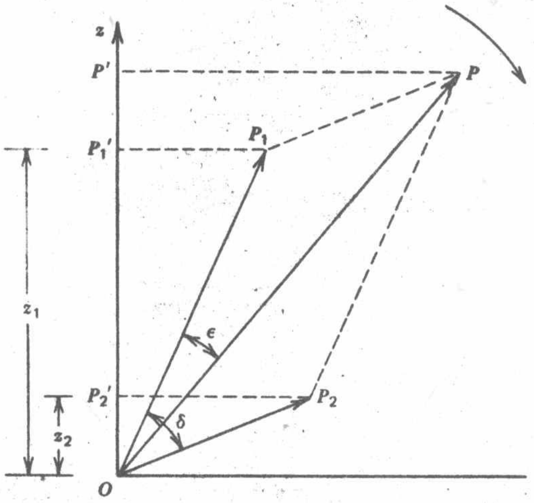


**Figure 2.11** Vector addition of two simple harmonic motions.

Let $A_1$ and $A_2$ be the amplitudes of two simple harmonic motions represented by $OP_1$ and $OP_2$, plotted from a fixed point $O$ (see Fig. 2.11). Let us suppose that the second motion, $OP_2$, is leading $OP_1$ and the phase angle between the two is $\delta$. Let us also suppose that each motion is rotating in a clockwise direction with a constant angular velocity $\omega = 2\pi/T$, where $T$ is the period for both motions.

The projections of $P_1$ and $P_2$ on the reference line ($Oz$) are $P'_1$ and $P'_2$, respectively. They describe the simple harmonic motions to be added. Both $P'_1$ and $P'_2$ have the same period $T$, while their amplitudes are $a_1$ and $a_2$, respectively. The second motion leads the first one by the phase angle $\delta$.

Both of the simple harmonic motions are on the same straight line $Oz$ and about the same point $O$, the resultant displacement at any time being the sum of two individual displacements $z_1$ and $z_2$. In Fig. 2.11, $z_1$ and $z_2$ are represented by $OP'_1$ and $OP'_2$.

If we now draw the parallelogram $OP_1OP_2$, from the parallelogram law we have $OP' = z$, which is the sum of $z_1$ and $z_2$ since $OP' = OP'_1 + OP'_2$. The amplitude of the resultant motion, $a$, is then represented by the length $OP$, since $OP'$ is the projection of the rotating vector $OP$ on the reference line.

We can also obtain the same result analytically instead of graphically. Let the combined motion be $a\cos(\omega t + \varepsilon)$. But

$$z = z_1 + z_2 = a_1 \cos \omega t + a_2 \cos(\omega t + \delta)$$

since

$$a_2 \cos(\omega t + \delta) = a_2 \cos \delta \cos \omega t - a_2 \sin \delta \sin \omega t$$

Therefore

$$\begin{aligned} \text{combined motion} &= (a_1 + a_2 \cos \delta) \cos \omega t \\ &\quad - a_2 \sin \delta \sin \omega t \\ &= a \cos(\omega t + \varepsilon) \\ &= a \cos \varepsilon \cos \omega t \\ &\quad - a \sin \varepsilon \sin \omega t \end{aligned}$$

or

$$a \cos \varepsilon = a_1 + a_2 \cos \delta \tag{2.8}$$

and

$$a \sin \varepsilon = a_2 \sin \delta \tag{2.9}$$

Squaring (2.8) and (2.9) and adding them, we have

$$\begin{aligned} a^2 &= (a_1 + a_2 \cos \delta)^2 + (a_2 \sin \delta)^2 \\ &= a_1^2 + a_2^2 \cos^2 \delta + 2a_1 a_2 \cos \delta + a_2^2 \sin^2 \delta \\ &= a_1^2 + a_2^2 + 2a_1 a_2 \cos \delta \end{aligned}$$

or

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos \delta} \tag{2.10}$$

Also, dividing (2.9) by (2.8) gives

$$\varepsilon = \tan^{-1}\left(\frac{a_2 \sin \delta}{a_1 + a_2 \cos \delta}\right) \tag{2.11}$$

where the resultant motion leads the lagging component (represented by $OP_1$) by the phase angle $\varepsilon$.

It is, therefore, shown that the graphical representation of this case is very suitable for determining the amplitude as well as the phase angle of the resultant motion. In addition, we have learned that the resultant motion of two harmonic motions is also harmonic and has the same frequency as the individual motions. Note, however, that the resultant motion is not a harmonic motion if the frequencies of the individual motions are not the same.

**Example 2.4**

A resultant motion is obtained by superimposing two displacements, namely,

$$z_1 = 8 \sin \omega t$$

$$z_2 = 7 \cos\left(\omega t - \frac{\pi}{4}\right)$$

Find the amplitude of the resultant motion and its phase angle in relation to the displacement of the first motion.

$$\begin{aligned} z_1 &= \text{the lagging component} \\ &= 8 \sin \omega t \\ &= 8 \cos\left(\frac{\pi}{2} - \omega t\right) = 8 \cos\left(\omega t - \frac{\pi}{2}\right) \end{aligned}$$