Mathematics Monograph Series 27

Singular Nonlinear Travelling Wave Equations: Bifurcations and Exact Solutions

Li Jibin (李继彬)

(奇非线性波方程：分支和精确解)

SCIENCE PRESS
Beijing

Mathematics Monograph Series 27

Singular Nonlinear Travelling Wave Equations: Bifurcations and Exact Solutions

(奇非线性波方程：分支和精确解)

Li Jibin(李继彬)

SCIENCE PRESS
Beijing

Responsible Editor: Zhao Yanchao

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, P. R. China

Printed in Beijing

ISBN 978-7-03-037917-7

Preface

Nonlinear wave phenomena are of great importance in the physical world and have been for a long time a challenging topic of research for both pure and applied mathematicians. There are numerous nonlinear evolution equations for which we need to analyze the properties of the solutions for time evolution of the systems.

The investigation of the travelling wave solutions to nonlinear evolution equations plays an important role in the mathematical physics. For example, the wave phenomena observed in fluid dynamics, plasma and elastic media are often medelled by the bell shaped solitary wave or kink shaped (wavefront) travelling wave solutions. To find exact travelling wave solutions for a given nonlinear wave system, since 1970's, a lot of methods have been developed such as the inverse scattering method, Darboux transformation method, Hirota bilinear method, algebraic-geometric method, tanh method, et al. (see (Ablowitz, 1991; Matveev & Salle, 1991; Hirota, 1971), et al.).

In 1993, Camassa and Holm used Hamiltonian methods to derive a new completely integrable dispersive shallow water wave equation, for which, they found that there exists a non-smooth peaked solitary wave solution (so called peakon). In (Rosenau & Hyman, 1994) and (Rosenau, 1997), the authors discussed "new wave mathematics" for some new integrable systems with dispersions (for example, $K(m, n)$ equation). So called "new wave", it is named by "peakon" "cuspon" and "compacton" et al., which are different from the bell shaped solitary wave solution. In his "concluding comments" of (Rosenau, 1997, pp318), for the understanding to the above mentioned nonanalytic wave (i.e. "new wave") solutions, the author stated that "unfortunately, as we have pointed (elsewhere in 1994's paper), a lack of proper mathematical tools makes this goal at the present time pretty much beyond our reach."

Fokas (1995) stated that peakons are peaked solitons, i.e., solitons with discontinuous first derivetive. Compactons are solitons with compact support. In order to answer the above mentioned "mathematical tools" problem, i.e., to solve the questions that how to understand the dynamics of the so-called compacton and peakon solutions? what is the reason of the smoothness change of travelling wave solutions? how do the travelling wave solutions depend on the change of parameters of the system? From 1998 to recent years, we have developed dynamical system method in (Li and Liu, 2000; Li, Wu & Zhu, 2006; Li & Chen 2007; Li & Dai, 2007; Li, et al., 2009), which provided rigorous mathematical understanding for these "new waves" to the singular nonlinear travelling wave systems.

Usually, the mathematical modeling of important phenomena arising in physics and biology often leads to integrable nonlinear wave equations(PDE). Generally, their travelling systems are ordinary differential equations (ODE). The studies of solitons and complete integrability of nonlinear wave equations and bifurcations, chaos of dynamical systems are two very active fields in nonlinear science. Because a homoclinic orbit of a travelling wave system (ODE) corresponds to a solitary wave solution of a nonlinear wave equation (PDE), while a heteroclinic orbit of a travelling wave system corresponds to a kink wave solution of a nonlinear wave equation. These relationships provide intersection points for the above two study fields. To consider travelling wave solutions of a partial differential equation, the essential work is to investigate the dynamical behavior of the corresponding ordinary differential equations (travelling wave systems). Therefore, the theory and method of dynamical systems play a pivotal role in the qualitative study of travelling wave solutions.

The aim of this book is to give a more systematic account for the bifurcation theory method of dynamical systems to find exact travelling wave solutions and their dynamics with an emphasis on the understanding of the above mentioned "new wave" for two classes of singular nonlinear travelling equations.

The materials of this book are completely taken from published papers written by the author, his collaborators and students. We hope that this book can serve as a guide to what can be cleared about the dynamical ideas in studying the travelling wave solutions of some nonlinear wave equations and for correcting some mistakes in understanding the dynamical behavior of some exact explicit travelling wave solutions.

Any reader trying to understand the subject of this book is only required to know the elementary theory of dynamical systems and elementary knowledge of mathematical and physical equations.

The publication of this book is supported by grant from the National Natural Science Foundation (10831003) of China and the research foundation of the center for dynamical systems and nonlinear studies of Zhejiang Normal University.

Li Jibin
March 10, 2013
Center for Dynamical Systems and Nonlinear
Studies, Zhejiang Normal University

Contents

Chapter 1

Some Physical Models Which Yield Two Classes of Singular Travelling Wave Systems

The mathematical modeling of important phenomena arising in physics and biology often leads to nonlinear wave equations. It is quite remarkable that many of these universal equations exhibit a regular behavior, typical of integrable partial differential systems (there exist Hamiltonian structures). And their travlling wave systems are also integrable ordinary differential equations, in which there exist some singular properties. In this chapter, we shall address some very interesting mathematical models which describe specific natural phenomena.

1.1 Nonlinear wave equations having the first class of singular nonlinear travelling wave systems

Many nonlinear wave models (partial differential equations) have their travelling wave systems in a form as follows:

$$\frac{d\phi}{d\xi} = y = \frac{1}{G^{\frac{2b}{a}}(\phi)}\frac{\partial H}{\partial y}, \quad \frac{dy}{d\xi} = -\frac{bG'(\phi)y^2 + F(\phi)}{aG(\phi)} = -\frac{1}{G^{\frac{2b}{a}}(\phi)}\frac{\partial H}{\partial \phi}, \tag{1.1.1}$$

where $\xi = x - ct$, a and b are real parameters,

$$H(\phi, y) = \frac{1}{2}y^2(G(\phi))^{\frac{2b}{a}} + \frac{1}{a}\int F(\phi)(G(\phi))^{\frac{2b}{a}-1}d\phi, \tag{1.1.2}$$

and the functions $G(\phi) \in C^2$, $F(\phi) \in C^1(-\infty, \infty)$ in order to guarantee the existence and uniqueness of the solutions of the initial value problem.

Suppose that there is $\phi = \phi_s$, such that $G(\phi_s) = 0$. Then, on the straight line $\phi = \phi_s$, the right hand of the second equation of system (1.1.1) is discontinuous. We say that system (1.1.1) is the first class of singular travelling wave systems. On the other hand, $\phi = \phi_s$ is a straight line solution of the system

$$\frac{d\phi}{d\zeta} = ayG(\phi), \quad \frac{dy}{d\zeta} = -(bG'(\phi)y^2 + F(\phi)), \tag{1.1.3}$$

where $d\xi = aG(\phi)d\zeta$, for $\phi \neq \phi_s$. System (1.1.3) is called the associated regular system of (1.1.1).

We next introduce some examples of nonlinear wave models which have the similar travelling wave systems as (1.1.1).

1.1.1 Higher order wave equations of Korteweg-De Vries type

In 1995, A.S. Fokas proposed to study a class of physically important integrable equations including higher order wave equations of the Korteweg-De Vries Type. Consider the motion of a two-dimensional, inviscid, incompressible fluid (water) lying above a horizontal flat bottom located at $y = -h_0$ (h_0 constant), and let there be air above the water. It turns out that, for such a system if the vorticity is zero initially, it remains zero. We analyze only such irrotational flows. This system is characterized by two parameters, $A = \dfrac{a}{h_0}$ and $B = \dfrac{h_0^2}{l^2}$, where a and l are typical values of the amplitude and of the wavelength of the waves. Let η and ϕ denote the position of the free surface and the velocity potential, respectively. Then $\eta(x,t)$ and $w(x,t)$ where $w = f_x$ and $\phi = \sum_0^{\infty}(-B)^m(1+A\eta)^{2m}f^{2m}/(2m)!$ (f^m denote the m-th derivative of f with respect to x) satisfy (see (Whitham, 1974))

$$\eta_t + w_x + A(\eta w)_x - \frac{1}{6}Bw_{xxx} - \frac{1}{2}AB(\eta w_{xx})_x - \frac{1}{2}A^2B(\eta^2 w_{xx})_x + O(B^2) = 0, \quad (1.1.4)$$

$$w_t + \eta_x + Aww_x + \frac{1}{2}B\eta_{xxx} + AB(\eta\eta_{xx} + w_x^2)_x + A^2B\left(2\eta^2 w_x^2 + \frac{1}{2}\eta^2\eta_{xx}\right)_x + O(B^2) = 0. \quad (1.1.5)$$

Suppose that $O(B)$ is less than $O(A)$ and the waves are unidirectional. Neglecting terms of $O(\alpha^4, \alpha^3\beta, \beta^2)$, equations (1.1.4) and (1.1.5) yield

$$\begin{aligned}&\eta_t + \eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} + \rho_1\alpha^2\eta^2\eta_x + \alpha\beta(\rho_2\eta\eta_{xxx} + \rho_3\eta_x\eta_{xx})\\ &\quad + \rho_4\alpha^3\eta^3\eta_x + \alpha^2\beta[\rho_5\eta^2\eta_{xxx} + \rho_6\eta\eta_x\eta_{xx} + \rho_7\eta_x^3] = 0,\end{aligned} \quad (1.1.6)$$

where $\alpha = \dfrac{3A}{2}, \beta = \dfrac{B}{6}, \rho_1 = -\dfrac{1}{6}, \rho_2 = \dfrac{5}{3}, \rho_3 = \dfrac{23}{6}, \rho_4 = \dfrac{1}{8}, \rho_5 = \dfrac{7}{18}, \rho_6 = \dfrac{79}{36}, \rho_7 = \dfrac{45}{36}$. Neglecting terms of $O(\alpha^2, \alpha\beta)$, equation (1.1.6) reduces to the KdV equation

$$\eta_t + \eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} = 0. \quad (1.1.7)$$

Neglecting terms of $O(\alpha^3, \alpha^2\beta)$, equation (1.1.6) reduces to the "more physically realistic form"

$$\eta_t + \eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} + \rho_1\alpha^2\eta^2\eta_x + \alpha\beta(\rho_2\eta\eta_{xxx} + \rho_3\eta_x\eta_{xx}) = 0. \quad (1.1.8)$$

We assume that $\rho_i, i = 1 \sim 7$ in (1.1.6) are considered as free parameters. Then, (1.1.8) and (1.1.6) are called the second order and the third order wave equations of KdV type, respectively.

A.S. Fokas also derived the following integrable generalizations of modified KdV equation:

$$u_t + u_x + \nu u_{xxt} + \beta u_{xxx} + \alpha u u_x + \frac{1}{3}\alpha\nu(u u_{xxx} + 2u_x u_{xx}) = 0 \tag{1.1.9}$$

and

$$\begin{aligned} &u_t + u_x + \nu u_{xxt} + \beta u_{xxx} + \alpha u u_x + \frac{1}{3}\alpha\nu(u u_{xxx} + 2u_x u_{xx}) + 3\mu\alpha^2 u^2 u_x \\ &+\nu\mu\alpha^2(u^2 u_{xxx} + u_x^3 + 4u u_x u_{xx}) + \nu^2\mu\alpha^2(u_x^2 u_{xxx} + 2u_x u_{xx}^2) = 0. \end{aligned} \tag{1.1.10}$$

First, we consider the travelling wave equation of the second order wave equations (1.1.8) of KdV type. Lett $\eta(x,t) = \phi(x-ct) = \phi(\xi)$, where c is the wave speed and $\xi = x - ct$, substituting $\phi(x-ct)$ into (1.1.8), we obtain

$$(1-c)\phi' + \frac{1}{2}\alpha(\phi^2)' + \beta\phi''' + \frac{1}{3}\alpha^2\rho_1(\phi^3)' + \alpha\beta(\rho_2(\phi\phi'')' + \frac{1}{2}(\rho_3 - \rho_2)((\phi')^2)') = 0, \tag{1.1.11}$$

where "$\prime$" is the derivative with respect to ξ. Integrating once with respect to ξ, we have the following travelling wave equation of (1.1.8)

$$\beta(1+\alpha\rho_2\phi)\phi'' + \frac{1}{2}\alpha\beta(\rho_3 - \rho_2)(\phi')^2 + \frac{1}{3}\alpha^2\rho_1\phi^3 + \frac{1}{2}\alpha\phi^2 + (1-c)\phi + g = 0, \tag{1.1.12}$$

where g is the integral constant. (1.1.12) is equivalent to the following two- dimensional system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\frac{3\alpha\beta(\rho_3-\rho_2)y^2 + 2\alpha^2\rho_1\phi^3 + 3\alpha\phi^2 + 6(1-c)\phi + g}{6\beta(1+\alpha\rho_2\phi)}. \tag{1.1.13}$$

We next assume that $g = 0$. Let $\rho_3 = p\rho_2$, where p is a real number. Then, for $\rho_3 \neq -2\rho_2,\ \rho_3 \neq \pm\rho_2$ i.e., $p \neq -2,\ p \neq \pm 1$, system (1.1.13) has the following first integrals

$$y^2 = h(1+\alpha\rho_2\phi)^{1-p} + \frac{A_0 + B_0\phi + C_0\phi^2 + D_0\phi^3}{3p(p+2)(p^2-1)\alpha^2\beta\rho_2^4}, \tag{1.1.14}$$

where

$$\begin{aligned} A_0 &= 6[\rho_2^2(1-c)(p+1)(p+2) + 2\rho_1 - (p+2)\rho_2], \quad B_0 = -\alpha(p-1)\rho_2 A_0, \\ C_0 &= 3p(p-1)\alpha^2\rho_2^2[2\rho_1 - (p+2)\rho_2], \quad D_0 = -2p\rho_1\alpha^3\rho_2^3(p^2-1); \end{aligned}$$

when $p = -2$, i.e., $\rho_3 = -2\rho_2$,

$$y^2 = h(1+\alpha\rho_2\phi)^3 - \frac{A_1 + B_1\phi + C_1\phi^2 + 6\rho_1(1 + 3\alpha\rho_2\phi + 3\alpha^2\rho_2^2\phi^2)\ln(1+2\alpha\rho_2\phi)}{9\beta\alpha^2\rho_2^4}, \tag{1.1.15}$$

where

$$A_1 = 11\rho_1 - 3\rho_2^2(1-c) - 3\rho_2, \quad B_1 = -9\alpha(\rho_2^3(1-c) + \rho_2^2 - 3\rho_1\rho_2),$$
$$C_1 = 9\alpha^2\rho_2^3(2\rho_1 - \rho_2);$$

when $p = -1$, i.e., $\rho_3 = -\rho_2$,

$$y^2 = h(1+\alpha\rho_2\phi)^2 - \frac{A_2 + B_2\phi + C_2\phi^2 + D_2\phi^3 + (E_2 + F_2\phi + G_2\phi^2)\ln(1+2\alpha\rho_2\phi)}{6\beta\alpha^2\rho_2^4}, \tag{1.1.16}$$

where

$$A_2 = -10\rho_1 - 6\rho_2(1-c) + 9\rho_2, \quad B_2 = \alpha(-8\rho_1\rho_2 - 12\rho_2^3(1-c) + 12\rho_2^2),$$
$$C_2 = 8\alpha^2\rho_1\rho_2^2, \quad D_2 = 4\alpha^3\rho_1\rho_2^3, \quad E_2 = 6\rho_2 - 12\rho_1,$$
$$F_2 = 12\alpha(\rho_2^2 - 2\rho_1\rho_2), \quad G_2 = 6\alpha^2(\rho_2^3 - 2\rho_1\rho_2^2);$$

when $p = 1$, i.e., $\rho_3 = \rho_2$,

$$(18\beta\alpha^2\rho_2^4)y^2 + \alpha\rho_2[4\alpha^2\rho_2^2\phi^3 + 3\alpha\rho_2(3\rho_2^2 - 2\rho_1)\phi^2 + (9\rho_2^2(1-c) - 18\rho_2 + 12\rho_1)\phi]$$
$$-6(6\rho_2^2(1-c) - 3\rho_2 - 2\rho_1)\ln(1+\alpha\rho_2\phi) = h, \tag{1.1.17}$$

where h is an arbitrary constant.

We see from (1.1.14) that if $1 - p = 2k$(k is an integer), or p is an irrational number, then we must consider the case $1 + \alpha\rho_2\phi > 0$, i.e., $\phi > \phi_s = -\dfrac{1}{\alpha\rho_2}$.

System (1.1.13) is a planar dynamical system defined in the 7-parameter space $(\alpha, \beta, c, \rho_1, \rho_2, \rho_3, g)$.

Second, we investigate the travlling wave equation of third order wave equations (1.1.6) of KdV type. Substituting $\eta = \phi(x - ct)$ into (1.1.6) and letting $y = \phi'(\xi)$, $z = \phi''(\xi)$, where "′" is the derivative with respect to ξ, we have the following 3-dimensional travelling wave system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = z,$$

$$\frac{dz}{d\xi} = -\frac{[\alpha\beta(\alpha\rho_7 y^2 + (\rho_3 + \alpha\rho_6\phi)z) + \alpha^3\rho_4\phi^3 + \alpha^2\rho_1\phi^2 + \alpha\phi + (1-c)]y}{\beta(1 + \alpha\rho_2\phi + \alpha^2\rho_5\phi^2)}. \tag{1.1.18}$$

There are two groups of parameter conditions (I) $\rho_6 = 2(\rho_5 + \rho_7)$ and (II) $\rho_7 = 0$ such that system (1.1.18) can be reduced to two 2-dimensional integrable systems.

We only consider the case (I). Then, we obtain from (1.1.18) that

$$(1-c)\phi' + \frac{1}{2}\alpha(\phi^2)' + \beta\phi''' + \frac{1}{3}\alpha^2\rho_1(\phi^3)' + \alpha\beta(\rho_2(\phi\phi'')' + \frac{1}{2}\alpha\beta(\rho_3 - \rho_2)((\phi')^2)'$$
$$+ \frac{1}{4}\alpha^3\rho_4(\phi^4)' + \alpha^2\beta(\rho_5(\phi^2\phi'')' + \rho_7(\phi(\phi')^2)') = 0. \tag{1.1.19}$$

Integrating once with respect to ξ, we have the following travelling wave equation of (1.1.6)

$$\beta(1+\alpha\rho_2\phi+\alpha^2\rho_5\phi^2)\phi''+\left(\frac{1}{2}\alpha\beta(\rho_3-\rho_2)+\alpha^2\beta\rho_7\phi\right)(\phi')^2$$

$$+\frac{1}{4}\alpha^3\rho_4\phi^4+\frac{1}{3}\alpha^2\rho_1\phi^3+\frac{1}{2}\alpha\phi^2+(1-c)\phi=0, \qquad (1.1.20)$$

where we take the integral constant $g=0$. (1.1.20) is equivalent to the following two-dimensional system

$$\frac{d\phi}{d\xi}=y,\quad \frac{dy}{d\xi}=-\frac{6\alpha\beta((\rho_3-\rho_2)+2\alpha\rho_7\phi)y^2+3\alpha^3\rho_4\phi^4+4\alpha^2\rho_1\phi^3+6\alpha\phi^2+12(1-c)\phi}{12\beta(1+\alpha\rho_2\phi+\alpha^2\rho_5\phi^2)}. \qquad (1.1.21)$$

Write that

$$S(\phi)=1+\alpha\rho_2\phi+\alpha^2\rho_5\phi^2, \qquad (1.1.22)$$

$$F(\phi)=f(\phi)\phi=(3\alpha^3\rho_4\phi^3+4\alpha^2\rho_1\phi^2+6\alpha\phi+12(1-c))\phi. \qquad (1.1.23)$$

Thus, (1.1.21) can be rewritten to the form

$$\frac{d\phi}{d\xi}=y,\quad \frac{dy}{d\xi}=-\frac{6\alpha\beta((\rho_3-\rho_2)+2\alpha\rho_7\phi)y^2+F(\phi)}{12\beta S(\phi)}. \qquad (1.1.24)$$

Clearly, system (1.1.21) is a planar dynamical system defined in the 10-parameter space $(\alpha,\beta,c,\rho_i), i=1-7$. Corresponding to different parameter subspace, it has different rich and complicated dynamical behavior.

The system (1.1.21) has a first integral for $\rho_2^2-4\rho_5>0$,

$$y^2S(\phi)^{\frac{\rho_7}{\rho_5}}\exp\left(A\text{artanh}\left(\frac{\rho_2+2\alpha\rho_5\phi}{\sqrt{\rho_2^2-4\rho_5}}\right)\right)$$

$$+\frac{1}{6\beta}\int S(\phi)^{\frac{\rho_7}{\rho_5}-1}F(\phi)\exp\left(-A\text{artanh}\left(\frac{\rho_2+2\alpha\rho_5\phi}{\sqrt{\rho_2^2-4\rho_5}}\right)\right)d\phi=h, \qquad (1.1.25)$$

where $A=\dfrac{2[\rho_5(\rho_2-\rho_3)+\rho_2\rho_7]}{\rho_5\sqrt{\rho_2^2-4\rho_5}}$; for $\rho_2^2-4\rho_5<0$,

$$y^2S(\phi)^{\frac{\rho_7}{\rho_5}}\exp\left(-iA\text{arctan}\left(\frac{\rho_2+2\alpha\rho_5\phi}{\sqrt{4\rho_5-\rho_2^2}}\right)\right)$$

$$+\frac{1}{6\beta}\int S(\phi)^{\frac{\rho_7}{\rho_5}-1}F(\phi)\exp\left(iA\text{arctan}\left(\frac{\rho_2+2\alpha\rho_5\phi}{\sqrt{4\rho_5-\rho_2^2}}\right)\right)d\phi=h; \qquad (1.1.26)$$

and for $\rho_2^2-4\rho_5=0$,

$$y^2(2+\alpha\rho_2\phi)^{\frac{8\rho_7}{\rho_2^2}}\exp\left(\frac{4[\rho_2(\rho_2-\rho_3)+4\rho_7]}{\rho_2^2(2+\alpha\rho_2\phi)}\right)$$

$$+\frac{2}{3\beta}\int F(\phi)(2+\alpha\rho_2\phi)^{\frac{8\rho_7-2\rho_2^2}{\rho_2^2}}\exp\left(\frac{4[\rho_2(\rho_2-\rho_3)+4\rho_7]}{\rho_2^2(2+\alpha\rho_2\phi)}\right)d\phi = h. \quad (1.1.27)$$

We see form (1.1.26) and (1.1.27) that to obtain an explicit integral formula for general parameters $\rho_i,\ i = 1 \sim 7$, it is very difficult.

Notice that the right hands of the second equations of (1.1.13),(1.1.21) are not continuous when

$$1 + \alpha\rho_2\phi = 0, \quad (1.1.28)$$

$$1 + \alpha\rho_2\phi + \alpha^2\rho_5\phi^2 = 0, \quad (1.1.29)$$

respectively. Therefore, we need to find the essential difference of the dynamical behavior between discontinuous systems and continuous systems.

1.1.2 Camassa-Holm equation and its generalization forms

In 1993, Camassa and Holm used Hamiltonian methods to derive a new completely integrable dispersive shallow water wave equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad (1.1.30)$$

where u is the fluid velocity in the x direction (or equivalently the height of the water's free surface above a flat bottom), k is a constant related to the critical shallow water wave speed, and subscripts denote partial derivatives. This equation retains higher order terms (the right-hand of (1.1.30)) in a small amplitude expansion of incompressible Euler's equations for unidirectional motion of waves at the free surface under the influence of gravity. Dropping these terms leads to the Benjamin-Bona-Mahoney (BBM) equation

$$u_t + u_x - u_{xxt} + uu_x = 0, \quad (1.1.31)$$

or at the same order, the KdV equation (1.1.7). Now, equation (1.1.30) is called Camassa-Holm (CH) equation.

In recent years, CH equation has been generalized to the following GCH equation:

$$u_t + 2ku_x - u_{xxt} + \frac{1}{2}[f(u)]_x = 2u_xu_{xx} + uu_{xxx}, \quad (1.1.32)$$

where $f(u)$ is a function of u. Specially, for $f(u) = \dfrac{2a}{m+1}u^{m+1}$, we have the so called modified Camassa-Holm (or mCH) equation

$$u_t + 2ku_x - u_{xxt} + au^mu_x = 2u_xu_{xx} + uu_{xxx}. \quad (1.1.33)$$

Dai & Huo (2000) derived the following far-field model equation

$$u_t + 3uu_x - u_{xxt} - \gamma(2u_xu_{xx} + uu_{xxx}) = 0 \quad (1.1.34)$$

for finite-length small-but-finite-amplitude waves in a prestretched rod composed of a compressible hyperelastic material when the material constant and prestress satisfy a given condition.

In 2001, Dullin et al. considered a generalized CH equation

$$u_t + c_0 u_x + 3uu_x - \alpha^2(u_{xxt} + uu_{xxx} + 2u_x u_{xx}) + \gamma u_{xxx} = 0, \tag{1.1.35}$$

which is called CH-γ equation. Here α,c_0 and γ are constants, and $\alpha \neq 0$. The CH-γ equation becomes the CH equation when $\alpha^2 = 1$, $c_0 = 2k$ and $\gamma = 0$.

In 1999, a new variant of (1.1.30) has been introduced by Degasperis and Procesi as

$$m_t + um_x + bu_x m = c_0 u_x - \gamma u_{xxx},$$

which is called the CH-DP equation. Here $m = u - \alpha^2 u_{xx}$ is a momentum variable, α, c_0, b, γ are constants and $\alpha \neq 0$. Clearly, the CH-DP equation can be rewritten as

$$u_t - c_0 u_x + (b+1)uu_x - \alpha^2(u_{xxt} + uu_{xxx} + bu_x u_{xx}) + \gamma u_{xxx} = 0. \tag{1.1.36}$$

Evidently, the CH-DP equation becomes the CH-γ equation when b=2. We see from (1.1.36) that CH-DP equation has one more parameter than CH-γ equation.

We look for travelling wave solutions of (1.1.36) in the form of $u(x,t) = \phi(x - ct) = \phi(\xi)$, where c is the wave speed and $\xi = x - ct$. Substituting $\phi(x - ct)$ into (1.1.36), we obtain

$$-(c + c_0)\phi' + (b+1)\phi\phi' - \alpha^2(\phi\phi''' + b\phi'\phi'') + (\alpha^2 c + \gamma)\phi''' = 0, \tag{1.1.37}$$

where "$'$" is the derivative with respect to ξ. Integrating (1.1.37) once with respect to ξ, we have the following travelling wave equation of (1.1.36)

$$-(c + c_0)\phi + \frac{1}{2}(b+1)\phi^2 - (\alpha^2\phi - \alpha^2 c - \gamma)\phi'' - \frac{1}{2}(b-1)\alpha^2(\phi')^2 + g = 0, \tag{1.1.38}$$

where g is the integral constant. (1.1.38) is equivalent to the following two-dimensional system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-\frac{1}{2}(b-1)\alpha^2 y^2 + \frac{1}{2}(b+1)\phi^2 - (c + c_0)\phi + g}{\alpha^2\phi - \alpha^2 c - \gamma}. \tag{1.1.39}$$

We next assume that $g = 0$. Then, system (1.1.39) has the following first integrals for $b \neq 0$, $b \neq 1$,

$$y^2 = h(\alpha^2\phi - \alpha^2 c - \gamma)^{1-b} + \frac{A_0 + B_0\phi + C_0\phi^2}{b(b-1)\alpha^6}, \tag{1.1.40}$$

where

$$A_0 = -2(\alpha^2 c + \gamma)(\alpha^2 c_0 - \gamma); \quad B_0 = -2\alpha^2(b-1)(\alpha^2 c_0 - \gamma); \quad C_0 = b(b-1)\alpha^4;$$

when $b=0$,

$$\begin{aligned} y^2 =& h(\alpha^2\phi-\alpha^2 c-\gamma) \\ &+\frac{A_1+B_1\phi+C_1\phi^2+2(\gamma-\alpha^2 c_0)(\alpha^2\phi-\alpha^2 c-\gamma)ln(\alpha^2\phi-\alpha^2 c-\gamma)}{\alpha^6}, \end{aligned} \tag{1.1.41}$$

where

$$A_1=\alpha^4 c(c+2c_0)+\gamma(2\alpha^2 c_0-\gamma), \quad B_1=\alpha^2(\alpha^2 c+\gamma), \quad C_1=\alpha^4;$$

when $b=1$,

$$y^2=h+\frac{A_2\phi+B_2\phi^2-2(\alpha^2 c+\gamma)(\alpha^2 c_0-\gamma)ln(\alpha^2\phi-\alpha^2 c-\gamma)}{\alpha^6}, \tag{1.1.42}$$

where $A_2=-2\alpha^2(\alpha^2 c_0-\gamma)$, $B_2=\alpha^4$. h is an arbitrary constant.

We see from (1.1.40) that if $1-b=2k$, (k is an integer) or b is an irrational number, then we must consider the case $\alpha^2\phi-\alpha^2 c-\gamma>0$, i.e., $\phi>\phi_s=\dfrac{\alpha^2 c+\gamma}{\alpha^2}$.

System (1.1.39) is a planar dynamical system defined in the 6-parameter space $(b,c,c_0,\alpha,\gamma,g)$.

Similarly, for the equation (1.1.32), we have its travelling wave equation

$$(\phi-c)\phi''=-\frac{1}{2}(\phi')^2+\frac{1}{2}f(\phi)+(2k-c)\phi+g, \tag{1.1.43}$$

where "$'$" is the derivative with respect to ξ. We write (1.3.14) as the differential equation system

$$\frac{d\phi}{d\xi}=y, \quad \frac{dy}{d\xi}=\frac{-y^2+2g+2(2k-c)\phi+f(\phi)}{2(\phi-c)}, \tag{1.1.44}$$

which has the following first integral

$$H(\phi,y)=(\phi-c)y^2-(2g\phi+(2k-c)\phi^2+F(\phi))=h, \tag{1.1.45}$$

where $F(\phi)=\int_0^\phi f(u)du$.

We next consider the following so called fully nonlinear Camassa-Holm equation

$$C(m,n,p): u_t+ku_x+\beta_1 u_{xxt}+\beta_2(u^m)_x+\beta_3 u_x(u^n)_{xx}+\beta_4 u(u^p)_{xxx}=0, \tag{1.1.46}$$

where $m,n,p\in Z^+$ and $k,\beta_i, i=1\sim 4$ are real parameters. Taking different m,n,p and k,β_i, this equation contains the above known equations as special examples. Let $u(x,t)=\psi(x-ct)=\psi(\xi)$. (1.1.46) reduces to

$$(k-c)\psi'-\beta_1 c\psi'''+\beta_2(\psi^m)'+\beta_3\psi'(\psi^n)''+\beta_4\psi(\psi^p)'''=0, \tag{1.1.47}$$