Lecture Notes in Mathematics

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on Homotopy Classification of Maps

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Introduction

The homotopy classification of maps, and closely related to it the study of extension and lifting problems, is a central topic in algebraic topology. Steenrod writes in [122]:

> "Many of the basic theorems of topology, and some of its most successful applications in other areas of mathematics, are solutions of particular extension problems. The deepest results of this kind have been obtained by the method of algebraic topology. The essence of the method is a conversion of the geometric problem into an algebraic problem which is sufficiently complex to embody the essential features of the geometric problem, yet sufficiently simple to be solvable by standard algebraic methods. Many extension problems remain unsolved, and much of the current development of algebraic topology is inspired by the hope of finding a truly general solution."

Obstruction theory is an attempt at such a general solution. This theory had its origins in the classical works of Hopf, Eilenberg, Steenrod and Postnikov around 1940 and has been developing ever since, albeit in an uncoordinated fashion. Portions of obstruction theory appear in most textbooks on algebraic topology, for instance in Steenrod's book on fiber bundles [120], or in the books by Spanier [116] and G.W. Whitehead [130]. These treatments often differ in approach and in the assumptions under which the theorems are proved, such as simply-connectedness, or that the fundamental group operate trivially, or that a fiber bundle be given instead of a fibration, or that only maps be considered instead of sections and retractions.

In this book we dispense with such restrictions wherever possible and so prove classical theorems in their full generality, for instance theorems on the Postnikov decomposition of a fibration, on primary and higher-order cohomology obstructions, and on the homotopy classification of maps that, as we show, apply to sections and retractions as well. Theorems of this kind are known to algebraic topologists, at least in a rough and ready way, and are commonly used. However, no self-contained exposition of obstruction theory has appeared.

We have here endeavored to give a systematic presentation of the subject, integrating the different approaches found in the literature. The essential tool for this is Eckmann-Hilton duality, which divides the presentation into two parts leading in parallel to the same goals. We have also systematized in another way by generalizing in stages. That is, we develop in 4 parallel stages the homotopy classification first of

1) maps using principal cofibrations,
2) sections of fibrations using principal cofibrations,
3) maps using relative principal cofibrations,
4) sections of fibrations using relative principal cofibrations

and then, as dual to these, the homotopy classification of

1') maps using principal fibrations,
2') retractions of cofibrations using principal fibrations,
3') maps using relative principal fibrations,
4') retractions of cofibrations using relative principal fibrations.

Stage 1) is a special case of 2) and 3), which are themselves special cases of 4). The reader need not shrink from having eight versions sung to him of the same old song, since in fact we develop only 2) and 2') thoroughly, in other stages often omitting details in explicitly formulating dual theorems, generalized ones, or their proofs. In the simpler stages we always point up the basic ideas clearly. We feel that the reader profits more from stagewise generalization of the theory, than if we had begun with the complicated versions 4) and 4') and only later moved on to the special cases, which are important in their own right.

In the literature most attention has been paid to approaches 1), 2) and 1'). We will show that classification theorems of Barcus-Barratt [5] for 1), and dually of James-Thomas [57] for 1'), are special cases of general classification results which we formulate using spectral sequences and which are valid for 2) and 2') also. Well-known in the context of 1) is the Puppe or cofiber sequence, as is in 1') the dual fiber sequence. We generalize these sequences at every stage to long exact classification sequences, and construct from them exact couples yielding spectral sequences for homotopy classification. These we will study in some detail. The classification sequences can also be derived from cofiber and fiber sequences in the category of ex-spaces, see (2.6).

However, we will construe them in terms of properties of primary obstructions and differences, which are already intimated in the classical works and have a natural significance in obstruction theory. Nonetheless, the importance of the ex-space category and relative methods as in 3), 4) and 3'), 4') is made clear by the existence of the principal reduction of CW-complexes and of Postnikov decompositions. These two existence proofs are a main result of this book. Relative methods in obstruction theory have been developed in the last ten years by James, Thomas, McClendon, Larmore, Becker and others. This book can be regarded as a systematic foundation of, and motivation for, these relative methods.

In the course of our presentation it was often necessary to introduce new notations because of the uncoordinated state of the theory in the literature. The text also contains various new, amplifying results. Cross-references to the literature for our results and definitions are contained under 'Remarks'. The bibliography is not intended however to encompass the entire subject.

I would like to acknowledge the support of the Sonderforschungsbereich 40 Theoretische Mathematik towards the completion of this book, in particular I am grateful to Mrs. Motee Spanier for typing the final version. I especially thank Stuart Clayton for translating the German manuscript.

H.J. Baues

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CHAPTER 6. ITERATED PRINCIPAL FIBRATIONS

CHAPTER 0. CONVENTIONS AND NOTATION

(0.0) Maps and homotopy. Excision theorems

The following conventions and notation will be maintained throughout the book. Let Top be the category of topological spaces and continuous maps, and let Top^o be the category of pointed topological spaces. Unless expressly stated to the contrary, from now on all spaces are pointed, that is, they all have distinguished base points * (starting with Chapter 1 we in fact require all spaces to be well-pointed, see (0.1.2)). Furthermore, all maps and all homotopies (denoted by $\simeq$) preserve the base points. The set of homotopy classes of base point-preserving maps $f: X \to Y$ will be denoted by $[X,Y]$. In this set 0 denotes the class of nullhomotopic maps. We denote by 1, 1_X or id the identity map. We will often use the same symbol to refer to a map and the homotopy class it represents. It will be clear from the context whether a map or a homotopy class is meant. The composition of maps or homotopy classes $f: X \to Y$ and $g: X \to Z$ will be denoted by $g \circ f$ or gf . Composition induces mappings of sets

$$g_* : [Y,X] \to [Y,Z] \quad \text{with} \quad g_*(f) = gf ,$$

$$f^* : [X,Z] \to [Y,Z] \quad \text{with} \quad f^*(g) = gf .$$

There are corresponding mappings for homotopy classes of maps between pairs.

Let $A \times B$ be the topological product of A and B , and let $A \vee B \subset A \times B$ be the one-point union or wedge of A and B , that is $A \vee B = A \times \{*\} \cup \{*\} \times B$. We have the canonical bijections

$$[A \vee B, X] \approx [A,X] \times [B,X]$$

$$[X, A \times B] = [X,A] \times [X,B] .$$

From pairs of maps we obtain maps $(f_1,f_2)\colon A \vee B \to X$ and $(g_1,g_2)\colon X \to A \times B$. The maps $c = (1,1)\colon X \vee X \to X$ and $d = (1,1)\colon X \to X \times X$ are called the folding map and the diagonal, respectively. We have

$$(f_1,f_2) = c \circ (f_1 \vee f_2) \quad \text{and} \quad (g_1,g_2) = (g_1 \times g_2) \circ d$$

Let X^Y be the space of non-pointed continuous maps $Y \to X$ with the compact-open topology. Then

(0.0.1) Exponential Law : For locally compact K the map

$$\vartheta : X^{K \times Y} \to (X^K)^Y$$

with $(f)(y)(t) = f(t,y)$ for $y \in Y$, $t \in K$ is a bijection. ϑ is a homeomorphism if Y and K are hausdorff, see [24].

Related to the exponential law are the following facts.

(a) Let K be locally compact. Then the evaluation map $X^K \times K \to X$ with $(f,t) \mapsto f(t)$ is continuous, see (4.14) of [24].

(b) Let K be locally compact and let $q\colon A \to B$ be an identification map. Then $q \times 1 : A \times K \to B \times K$ is also an identification map, see (4.13) of [24].

(c) As an extension of (b), let $A \subset X$ and let A be compact and let $q\colon X \to X/A$ be the identification map. Then for any space Z the map $q \times 1 : X \times Z \to (X/A) \times Z$ is also an identification map.

(d) Let $i\colon A \to B$ be an embedding. Then for any space Z the map $i^Z\colon A^Z \to B^Z$ is an embedding, see 4.6 of [24].

(e) In connection with (c) we can say the following. Let the map $p\colon A \to B$ be surjective and such that for every compact subset

L of B there is a compact subset $K \subset A$ with $p(K) = L$. Then for any space Z the map $Z^p: Z^B \longrightarrow Z^A$ with $f \mapsto f \circ p$ is an embedding. An example of such a map p is the map q in (c).

(f) For Z hausdorff and arbitrary X, Y we have

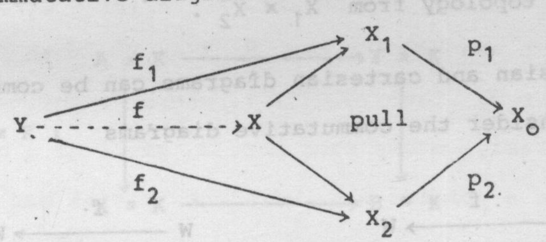
$$(X \times Y)^Z = X^Z \times Y^Z$$

(g) For arbitrary X, Y, Z we have

$$X^{Y+Z} = X^Y \times X^Z$$

where Y+Z is the topological sum, that is, the disjoint union.

In the commutative diagram of unbroken arrows

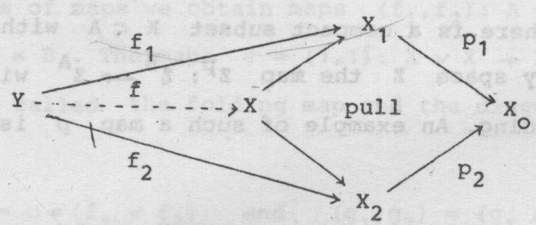


the subdiagram 'push' is called a cocartesian square or <u>pushout</u> when to every pair of maps f_1, f_2 there exists exactly one map $f = f_1 \cup f_2$ extending the diagram. By 'extending the diagram' we always mean 'commutatively'. Such an X is uniquely determined up to homeoporphism. There exists a cocartesian square for i_1 and i_2, since we can take

$$X = X_1 \cup_{X_o} X_2 = (X_1 + X_2)/\sim$$

where the equivalence relation in the disjoint union $X_1 + X_2$ is generated by $i_1(x) \sim i_2(x)$ with $x \in X_o$. X is given the quotient topology. If i_1 is an inclusion, X is called an adjunction space.

There is a dual definition to the preceeding one. In the commutative diagram

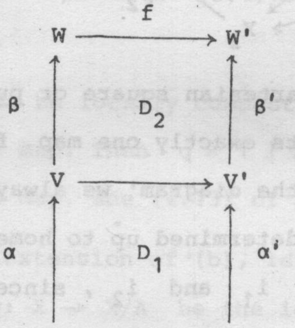


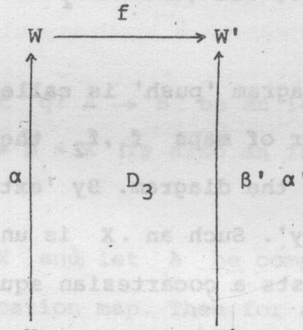
the subdiagram 'pull' is called a cartesian square or <u>pullback</u> when to every pair of maps f_1, f_2 there exists exactly one map $f = f_1 \underline{\times} f_2$ extending the diagram. Such an X is uniquely determined up to homeomorphism. There exists a cartesian square for p_1 and p_2, since we can take

$$X = X_1 \times_{X_o} X_o = \{(x,y) \in X_1 \times X_2 \mid p_1(x) = p_2(y)\}$$

with the subspace topology from $X_1 \times X_2$.

(O.O.2) Cocartesian and cartesian diagrams can be combined in the following way. Consider the commutative diagrams





If D_1 is cocartesian, then

D_2 is cocartesian $\Leftrightarrow$ D_3 is cocartesian.

If D_2 is cartesian, then

$$D_1 \text{ is cartesian} \Leftrightarrow D_3 \text{ is cartesian}$$

(0.0.3) Cocartesian and cartesian squares are compatible with products and mapping spaces in the following ways. Let a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & (D) & \downarrow \\ X & \longrightarrow & B \end{array}$$

be given. Then

(I) If (D) is cocartesian and K is locally compact, the diagram

$$(D) \times K : \quad \begin{array}{ccc} A \times K & \longrightarrow & Y \times K \\ \downarrow & & \downarrow \\ X \times K & \longrightarrow & B \times K \end{array}$$

is also cocartesian, as follows easily from (0.0.1)(b). An extension of this will be described in (0.5.3).

(II) If (D) is cocartesian and X, Y, B are compact hausdorff (or alternatively if the quotient map $X+Y \to B$ satisfies the condition in (0.0.1)(e)), the diagram

$$Z^{(D)} : \quad \begin{array}{ccc} Z^A & \longleftarrow & Z^Y \\ \uparrow & & \uparrow \\ Z^X & \longleftarrow & Z^B \end{array}$$

is cartesian, as follows easily from (0.0.1)(e) and (g) .

(III) If (D) is cartesian and Z is hausdorff, the diagram

$$(D)^Z : \quad \begin{array}{ccc} A^Z & \longrightarrow & Y^Z \\ \downarrow & & \downarrow \\ X^Z & \longrightarrow & B^Z \end{array}$$

is also cartesian, as follows easily from (0.0.1)(d) and (f).

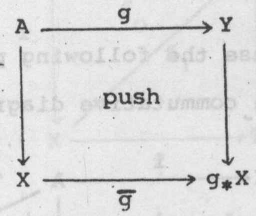
We now define some further homotopy concepts. Let $I = [0,1]$ be the unit interval and let $H: f_o \simeq f_1 : X \longrightarrow Y$ be a homotopy. That is, $H: I \times X \longrightarrow Y$ is a map with $H_o = f_o$ and $H_1 = f_1$, where for $t \in I$ the pointed map $H_t: X \longrightarrow Y$ is defined by $H_t(x) = H(t,x)$ for $x \in X$. The map H gives us the adjoint map $\bar{H}: X \longrightarrow Y^I$ with $H(x)(t) = H(t,x)$ $q_o\bar{H} = f_o$ and $q_1\bar{H} = f_1$ (see (0.0.1)), where we define $q_t(\sigma) = \sigma(t)$ for $\sigma \in Y^I$. Conversely, every such map H gives us a homotopy $H: f_o \simeq f_1$. H is a pointed map, with the trivial map $0 \in Y^I$ as base point in Y^I .

Given the maps (in Top^o)

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & Y \\ {\scriptstyle i}\downarrow & \nearrow {\scriptstyle g_o, g_1} & \\ X & & \end{array}$$

we call $H: g_o \simeq g_1$ a <u>homotopy under</u> A when for all $t \in I$ we have $H_t \circ i = g$. The set of homotopy classes under A is denoted by $[X,Y]^g$ or $[X,Y]^A$. It will also be referred to as the homotopy set relative g , especially when i is an inclusion. If g is the identity, the homotopy set under A will also be called the <u>retraction homotopy set</u>

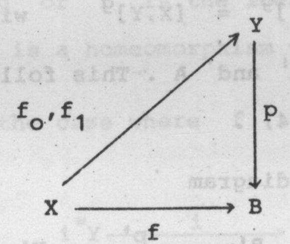
for i , denoted by $\langle X,A\rangle$. Every homotopy set under A can be regarded as a retraction homotopy set in the following way. Let



be the cocartesian diagram for (i,g) . Then $\overline{g}$ induces a bijection

(O.O.4) $$[X,Y]^g = \langle g_*X,Y\rangle$$

Dual to 'homotopy under' is the concept of 'homotopy over', defined as follows. Given the maps (in Top^o)



we call $H: f_o \simeq f_1$ a <u>homotopy over</u> B when for all $t \in I$ we have $p \circ H_t = f$. The set of homotopy classes over B is denoted by $[X,Y]_f$ or $[X,Y]_B$. It will also be referred to as the homotopy set of liftings of f , especially when p is a fibration. If f is the identity, the homotopy set over B will be called the <u>section homotopy set</u> for p , denoted by $\langle B,Y\rangle$. It will always be clear from the context whether $\langle\ ,\ \rangle$ denotes a section homotopy set or a retraction homotopy set. Every homotopy set over B can be regarded as a section homotopy set as follows. Let