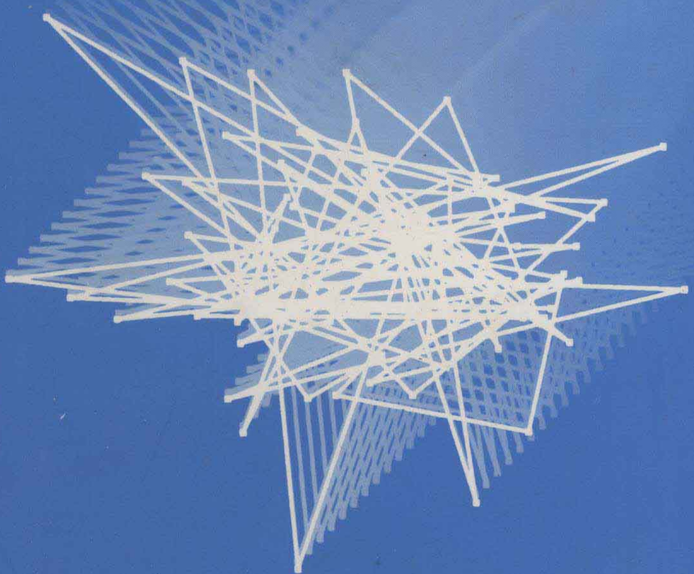

STOCHASTIC DIFFERENTIAL EQUATIONS IN SCIENCE AND ENGINEERING

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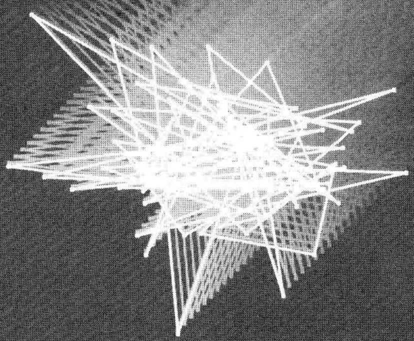

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STOCHASTIC DIFFERENTIAL EQUATIONS IN SCIENCE AND ENGINEERING

To Rose-Marie Henderson

A good friend and spouse

PREFACE

This book arose from a friendship formed when we were both faculty members of the Department of Physics, Universidad Autonoma Metropolitana, Iztapalapa Campus, in Mexico City. Plaschko was teaching an intermediate to advanced course in mathematical physics. He had written, with Klaus Brod, a book entitled, "Hoehere Mathematische Methoden fuer Ingenieure und Physiker", that Henderson admired and suggested that be translated into English and be updated and perhaps expanded somewhat.

However, we both prefer new projects and this suggested instead that a book on Stochastic Differential Equations be written and this project was born. This is an important emerging field. From its inception with Newton, physical science was dominated by the idea of determinism. Everything was thought to be determined by a set of second order differential equations, Newton's equations, from which everything could be determined, at least in principle, if the initial conditions were known. To be sure, an actual analytic solution would not be possible for a complex system since the number of dynamical equations would be enormous; even so, determinism prevailed. This idea took hold even to the point that some philosophers began to speculate that humans had no free will; our lives were determined entirely by some set of initial conditions. In this view, even before the authors started to write, the contents of this book were determined by a set of initial conditions in the distant past. Dogmatic Marxism endorsed such ideas, although perhaps not so extremely.

Deterministic Newtonian mechanics yielded brilliant successes. Most astronomical events could be predicted with great accuracy.

Even in case of a few difficulties, such as the orbit of Mercury, Newtonian mechanics could be replaced satisfactorily by equally deterministic general relativity. A little more than a century ago, the case for determinism was challenged. The seemingly random motion of the Brownian motion of suspended particles was observed as was the sudden transition of the flow of a fluid past an object or obstacle from lamanar flow to chaotic turbulence. Recent studies have shown that some seemingly chaotic motion is not necessarily inconsistent with determinism (we can call this quasi-chaos). Even so, such problems are best studied using probablistic notions. Quantum theory has shown that the motion of particles at the atomic level is fundamentally nondeterministic. Heisenberg showed that there were limits to the precision with which physical properties could be determined. One can only assign a probablity for the value of a physical quantity. The consequence of this idea can be manifest even on a macroscopic scale. The third law of thermodynamics is an example.

Stochastic differential equations, the subject of this monograph, is an interesting extension of the deterministic differential equations that can be applied to Brownian motion as well as other problems. It arose from the work of Einstein and Smoluchowski among others. Recent years have seen rapid advances due to the development of the calculii of Ito and Stratonovich.

We were both trained as mathematicians and scientists and our goal is to present the ideas of stochastic differential equations in a short monograph in a manner that is useful for scientists and engineers, rather than mathematicians and without overpowering mathematical rigor. We presume that the reader has some, but not extensive, knowledge of probability theory. Chapter 1 provides a reminder and introduction to and definition of some fundamental ideas and quantities, including the ideas of Ito and Stratonovich. Stochastic differential equations and the Fokker–Planck equation are presented in Chapters 2 and 3. More advanced applications follow in Chapter 4. The book concludes with a presentation of some numerical routines for the solution of ordinary stochastic differential equations. Each chapter contains a set of exercises whose purpose is to aid the reader in understanding the material. A CD-ROM that provides

MATHEMATICA and FORTRAN programs to assist the reader with the exercises, numerical routines and generating figures accompanies the text.

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June, 2006

INTRODUCTION

The theory of deterministic chaos has enjoyed during the last three decades a rapidly increasing audience of mathematicians, physicists, engineers, biologists, economists, etc. However, this type of "chaos" can be understood only as quasi-chaos in which all states of a system can be predicted and reproduced by experiments.

Meanwhile, many experiments in natural sciences have brought about hard evidence of stochastic effects. The best known example is perhaps the Brownian motion where pollen submerged in a fluid experience collisions with the molecules of the fluid and thus exhibit random motions. Other familiar examples come from fluid or plasma dynamic turbulence, optics, motions of ions in crystals, filtering theory, the problem of optimal pricing in economics, etc. The study of stochasticity was initiated in the early years of the 1900's. Einstein [1], Smoluchowsky [2] and Langevin [3] wrote pioneering investigations. This work was later resumed and extended by Ornstein and Uhlenbeck [4]. But investigation of stochastic effects in natural science became more popular only in the last three decades. Meanwhile studies are undertaken to calculate or at least approximate the effect of stochastic forces on otherwise deterministic oscillators, to investigate the stability or the transition to stochastic chaos of the latter oscillator.

To motivate the following considerations of stochastic differential equations (SDE) we introduce a few examples from natural sciences.

(a) Pendulum with Stochastic Excitations

We study the linearized pendulum motion $x(t)$ subjected to a stochastic effect, called white noise

$$\ddot{x} + x = \beta \xi_t,$$

where β is an intensity constant, t is the time and ξ_t stands for the white noise, with a single frequency and constant spectrum. For $\beta = 0$ we obtain the homogeneous deterministic (non-stochastic) traditional pendulum motion. We can expect that the stochastic effect disturbs this motion and destroys the periodicity of the motion in the phase space $(x, \dot{x})$. The latter has closed solutions called limit cycles. It is an interesting task to investigate whether the solutions disintegrate into scattered points (stochastic chaos). We will cover this problem later in Section 2.3 and find that the average motion (in a sense to be defined in Section 1.2 of Chapter 1) of the pendulum is determined by the deterministic limit ($\beta = 0$) of the stochastic pendulum equation.

(b) Stochastic Growth of Populations

$\mathrm{N}(t)$ is the number of the members of a population at the time t, α is the constant of the deterministic growth and β is again a constant characterizing the intensity of the white noise. Thus we study the growth problem in terms of the linear scenario

$$\frac{\mathrm{dN}}{\mathrm{d}t} = \alpha\mathrm{N} + \beta\mathrm{N}\xi_t.$$

The deterministic limit ($\beta = 0$) of this equation describes the growth of a population living on an unrestricted area with unrestricted food supply. Its solution (the number of such a population) grows exponentially. The stochastic effects, or the white noise describes a stochastic varying food supply that influences the growth of the population. We will consider this problem in the Section 2.1.1 and find again that the average of the population is given by the deterministic limit.

(c) Diffraction of Optical Waves

The transfer function $\mathrm{T}(\omega)$; $\omega = (\omega_1, \omega_2)$ of a two-dimensional optical device is defined by

$$\mathrm{T}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \mathrm{F}(x, y)\mathrm{F}^*(x - \omega_1, y - \omega_2)/\mathrm{N};$$

$$\mathrm{N} = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y |\mathrm{F}(x, y)|^2,$$

where F is a complex wave amplitude and $F^* = \mathrm{cc}(F)$ is its complex conjugate. The parameter N denotes the normalization of $|F(x, y)|^2$ and the variables x and y stand for the coordinates of the image plane. In a simplified treatment, we assume that the wave form is given by

$$F = |F| \exp(-ik\Delta); \quad |F|, k = \text{const},$$

where k and Δ stand for the wave number and the phase of the waves, respectively. We suppose that the wave emerging from the optical instrument (e.g. a lens) exhibits a phase with two different deviations from a spherical structure $\Delta = \Delta_c + \Delta_r$ with a controlled or deterministic phase $\Delta_c(x, y)$ and a random phase $\Delta_r(x, y)$ that arises from polishing the optical device or from atmospheric influences. Thus, we obtain

$$T(\omega) = \frac{1}{K} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp\{ik[\Delta(x - \omega_1, y - \omega_2) - \Delta(x, y)]\},$$

where K is used to include the normalization. In simple applications we can model the random phase using white noise with a Gaussian probability density. To evaluate the average of the transfer function $\langle T(\omega) \rangle$ we need to calculate the quantity

$$\langle \exp\{ik[\Delta_r(x - \omega_1, y - \omega_2) - \Delta_r(x, y)]\}\rangle.$$

We will study the Gaussian probability density and complete the task to determine the average written in the last line in Section 1.3 of Chapter 1. An introduction to random effects in optics can be found in O'Neill [5].

(d) Filtering Problems

Suppose that we have performed experiments of a stochastic problem such as the one in (a) in an interval $t \in [0, u]$ and we obtain as result say $A(v), v = [0, u]$. To improve the knowledge about the solution we repeat the experiments for $t \in [u, T]$ and we obtain $A(t), t = [u, T]$. Yet due to inevitable experimental errors we do not obtain $A(t)$ but a result that includes an error $A(t)$ + 'noise'. The question is now how can we filter the noise away? A filter is thus, an instrument to clean a result and remove the noise that arises during the observation. A typical problem is where a signal with unknown frequency is transmitted (e.g. by an electronic device) and it suffers during the transmission the addition of a noise. If the transmitted signal is stochastic itself (as in the case of music) we need to develop a non-deterministic model for the signal with the aid of a stochastic differential equation. To study basic the ideas of filtering problems the reader in referred to the book of Stremler [6].

(e) Fluidmechanical Turbulence

This is the perhaps most challenging and most intricate application of statistical science. We consider here the continuum dynamics of a flow field influenced by stochastic effects. The latter arise from initial conditions (e.g. at the nozzle of a jet flow, or at the entry region of a channel flow) and/or from background noise (e.g. acoustic waves). In the simplest case, the incompressible two-dimensional flows, there are three characteristic variables (two velocity components and the pressure). These variables are governed by the Navier–Stokes equations (NSEs). The latter are a set of three nonlinear partial differential equations that included a parameter, the Reynolds number R. The inverse of R is the coefficient of the highest derivatives of the NSEs. Since turbulence occurs at intermediate to high values of the R, this phenomenon is the rule and not the exception in Fluid Dynamics and it occurs in parameter regions where the NSEs are singular. Nonlinear SDEs — such as the NSEs — lead additionally to the problem of the closure, where the equation governing the statistical moment of nth order contains moments of the $(n+1)$th order.

Hopf [7] was the first to try to find a theoretical approach to solve the problem for the idealized case of isotropic homogenous turbulence, a flow configuration that can be approximately realized in grid flows. Hopf assumed that the turbulence is Gaussian, an assumption that facilitates the calculation of higher statistical moments of the distribution (see Section 1.3 in Chapter 1). However, later measurements showed that the assumption of a Gaussian distribution was rather unrealistic. Kraichnan [8] studied the problem again in the 60's and 70's with the direct triad interaction theory in the idealized configuration of homogeneous isotropic turbulence. However, this rather involved analysis could only be applied to calculate the spectrum of very small eddies where the viscosity dominates the flow. Somewhat more progress has been achieved by the investigation of Rudenko and Chirin [9]. The latter predicted with aid of stochastic initial conditions with random phases a broad banded spectra of a nonlinear model equation. During the last two decades there was the intensive work done to investigate the Burgers equation and this research is summarized in part by Wojczinsky [10]. The Burgers equation is supposed to be a reasonable one-dimensional model of the NSEs. We will give a short account on the work done in [9] in Chapter 4.

GLOSSARY

AC	almost certainly
BC	boundary condition
$\mathrm{dB}_t = \mathrm{dW}_t = \xi_t \mathrm{d}t$	differential of the Brownian motion (or equivalently Wiener process)
$\mathrm{cc}(a) = a^*$	complex conjugate of a
D	dimension or dimensional
DF	distribution function
DOF	degrees of freedom
δ_{ij}	Kronecker delta function
$\delta(x)$	Dirac delta function
EX	exercise at the end of a chapter
FPE	Fokker–Planck equation
$\Gamma(x)$	gamma function
GD	Gaussian distribution
GPD	Gaussian probability distribution
HPP	homogeneous Poisson process
$\mathrm{H}_n(x)$	Hermite polynomial of order n
IC	initial condition
IID	identically independently distributed

IFF	if and only if
IMSL	international mathematical science library
$\mathcal{L}$	Laplace transform
M	master, as in master equation
MCM	Monte Carlo method
NSE	Navier–Stokes equation
NIGD	normal inverted GD
$\mathrm{N}(\mu, \sigma)$	normal distribution with μ as mean and σ as variance
o	Stratonovich theory
ODE	ordinary differential equation
PD	probability distribution
PDE	partial differential equation
PDF	probability distribution function
PSDE	partial SDE
r	Reynolds number
RE	random experiment
RN	random number
RV	random variable
$\mathrm{Re}(a)$	real part of a complex number
R, C	sets of real and complex numbers, respectively
S	Prandt number
SF	stochastic function
SI	stochastic integral
SDE	stochastic differential equation
SLNN	strong law of large numbers

TPT	transition probability per unit time
WP	Wiener process
WS	Wiener sheet
WKB	Wentzel, Kramers, Brillouin
WRT	with respect to
$\mathrm{W}(t)$	Wiener white (single frequency) noise
$\langle a \rangle$	average of a stochastic variable a
$\sigma^2 = \langle a^2 \rangle - \langle a \rangle \langle a \rangle$	variance
$\langle x \vert y \rangle, \langle x, u \vert y, v \rangle$	conditional averages
$s \wedge t$	minimum of s and t
$\forall$	for all values of
$\in$	element of
$\int f(x)\mathrm{d}x$	short hand for $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$
♣	end of an example
□	end of definition
✾	end of theorem

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