Elements of the Representation Theory of Associative Algebras

2: Tubes and Concealed Algebras of Euclidean type

DANIEL SIMSON
and ANDRZEJ SKOWROŃSKI

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Volume 2 Tubes and Concealed Algebras of Euclidean type

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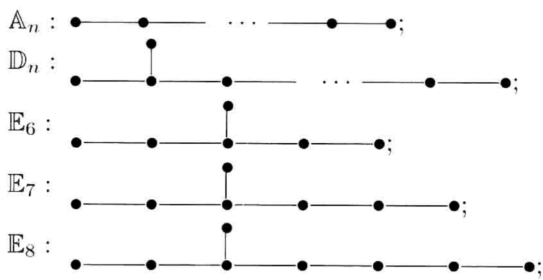
To our Wives

Sabina and Mirosława

Introduction

The first volume serves as a general introduction to some of the techniques most commonly used in representation theory. The quiver technique, the Auslander–Reiten theory and the tilting theory were presented with some application to finite dimensional algebras over a fixed algebraically closed field.

In particular, a complete classification of those hereditary algebras that are representation-finite (that is, admit only finitely many isomorphism classes of indecomposable modules) is given. The result, known as Gabriel's theorem, asserts that a basic connected hereditary algebra A is representation-finite if and only if the quiver Q_A of A is a Dynkin quiver, that is, the underlying non-oriented graph $\overline{Q}_A$ of Q_A is one of the Dynkin diagrams



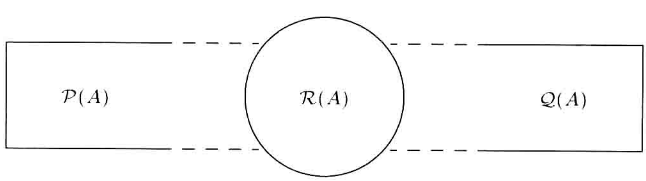
(n vertices, $n \geq 1$);

(n vertices, $n \geq 4$);

We also study in Volume 1 the class of hereditary algebras that are representation-infinite. It is shown in Chapter VIII that if B is a representation-infinite hereditary algebra, or B is a tilted algebra of the form

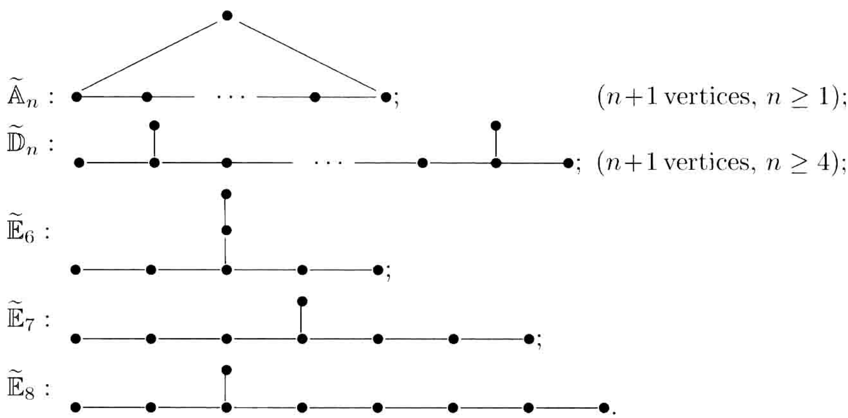
$$B = \operatorname{End} T_{KQ},$$

where KQ is a representation-infinite hereditary algebra and T_{KQ} is a postprojective tilting KQ-module, then B is representation-infinite and the Auslander–Reiten quiver $\Gamma(\operatorname{mod} B)$ of B has the shape



where $\operatorname{mod} B$ is the category of finite dimensional right B-modules, $\mathcal{P}(B)$ is the unique postprojective component of $\Gamma(\operatorname{mod} B)$ containing all the indecomposable projective B-modules, $\mathcal{Q}(B)$ is the unique preinjective component of $\Gamma(\operatorname{mod} B)$ containing all the indecomposable injective B-modules, and $\mathcal{R}(B)$ is the (non-empty) regular part consisting of the remaining components of $\Gamma(\operatorname{mod} B)$.

A prominent rôle in the representation theory is played by the class of hereditary algebras that are representation-infinite and minimal with respect to this property. They are just the hereditary algebras of Euclidean type, that is, the path algebras KQ, where Q is a connected acyclic quiver whose underlying non-oriented graph $\overline{Q}$ is one of the following Euclidean diagrams



It is shown in Chapter VII that the underlying graph $\overline{Q}$ of a finite connected quiver $Q = (Q_0, Q_1)$ is a Dynkin diagram, or a Euclidean diagram, if and only if the associated quadratic form $q_Q : \mathbb{Z}^{|Q_0|} \longrightarrow \mathbb{Z}$ is positive definite, or positive semidefinite and not positive definite, respectively.

The main aim of Volumes 2 and 3 is to study the representation-infinite tilted algebras $B = \operatorname{End} T_{KQ}$ of a Euclidean type Q and, in particular, to give a fairly complete description of their indecomposable modules, their module categories $\operatorname{mod} B$, and the Auslander–Reiten quivers $\Gamma(\operatorname{mod} B)$.

For this purpose, we introduce in Chapter X a special type of components in the Auslander–Reiten quivers of algebras, namely stable tubes, and study their behaviour in module categories. In particular, we present a handy criterion on the existence of a standard self-hereditary stable tube, due to Ringel [215], and a characterisation of generalised standard stable tubes, due to Skowroński [246], [247], [254].

In Chapters XI and XII, we present a detailed description and properties of the regular part $\mathcal{R}(B)$ of the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, B)$ of any concealed algebra B of Euclidean type, that is, a tilted algebra

$$B = \mathrm{End}\, T_{KQ}$$

of a Euclidean type Q defined by a postprojective tilting KQ-module T_{KQ}.
In particular, it is shown that:

- the regular part $\mathcal{R}(B)$ of the Auslander–Reiten quiver $\Gamma(\mathrm{mod} B)$ is a disjoint union of the $\mathbb{P}_1(K)$-family

$$\boldsymbol{\mathcal{T}}^B = \{\mathcal{T}_\lambda^B\}_{\lambda \in \mathbb{P}_1(K)}$$

  of pairwise orthogonal standard stable tubes $\mathcal{T}_\lambda^B$, where $\mathbb{P}_1(K)$ is the projective line over K,
- the family $\boldsymbol{\mathcal{T}}^B$ separates the postprojective component $\mathcal{P}(B)$ from the preinjective component $\mathcal{Q}(B)$,
- the module category $\mathrm{mod} B$ is controlled by the Euler quadratic form $q_B : K_0(B) \longrightarrow \mathbb{Z}$ of the algebra B.

A crucial rôle in the investigation is played by the canonical algebras of Euclidean type, introduced by Ringel [215]. As an application of the developed theory, we present in Chapter XIII a complete list of indecomposable regular KQ-modules over any path algebra KQ of a canonically oriented Euclidean quiver Q, and we show how a simple tilting process allows us to construct the indecomposable regular modules over any path algebra KQ of a Euclidean type Q.

In Chapter XIV, we give the Happel–Vossieck [112] characterisation of the minimal representation-infinite algebras B having a postprojective component in the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, B)$. As a consequence, we get a finite representation type criterion for algebras. We also present a complete classification, by means of quivers with relations, of all concealed algebras of Euclidean type, due independently by Bongartz [29] and Happel–Vossieck [112].

In Volume 3, we introduce some concepts and tools that allow us to give there a complete description of arbitrary representation-infinite tilted algebras B of Euclidean type and the module category $\mathrm{mod}\, B$, due to Ringel [215]. We also investigate the wild hereditary algebras $A = KQ$, where Q is an acyclic quiver such that the underlying graph is neither a Dynkin nor a Euclidean diagram. We describe the shape of the components of the regular part $\mathcal{R}(A)$ of $\Gamma(\mathrm{mod}\, A)$ and we establish a wild behaviour of the category $\mathrm{mod}\, A$, for any such an algebra A. Finally, we introduce in Volume 3 the concepts of tame representation type and of wild representation type for algebras, and we discuss the tame and the wild nature of module categories mod B. Also, we present (without proofs) selected results of the representation theory of finite dimensional algebras that are related to the material discussed in the book.

It was not possible to be encyclopedic in this work. Therefore many important topics from the theory have been left out. Among the most notable omissions are covering techniques, the use of derived categories and partially ordered sets. Some other aspects of the theory presented here are discussed in the books [10], [15], [16], [91], [121], [235], and especially [215].

We assume that the reader is familiar with Volume 1, but otherwise the exposition is reasonably self-contained, making it suitable either for courses and seminars or for self-study. The text includes many illustrative examples and a large number of exercises at the end of each of the Chapters X-XIV.

The book is addressed to graduate students, advanced undergraduates, and mathematicians and scientists working in representation theory, ring and module theory, commutative algebra, abelian group theory, and combinatorics. It should also, we hope, be of interest to mathematicians working in other fields.

Throughout this book we use freely the terminology and notation introduced in Volume 1. We denote by K a fixed algebraically closed field. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ mean the sets of natural numbers, integers, rational, real, and complex numbers. The cardinality of a set X is denoted by $|X|$. Given a finite dimensional K-algebra A, the A-module means a finite dimensional right A-module. We denote by $\operatorname{Mod} A$ the category of all right A-modules, by $\operatorname{mod} A$ the category of finite dimensional right A-modules, and by $\Gamma(\operatorname{mod} A)$ the Auslander–Reiten translation quiver of A. The ordinary quiver of an algebra A is denoted by Q_A. Given a matrix $C = [c_{ij}]$, we denote by C^t the transpose of C.

A finite quiver $Q = (Q_0, Q_1)$ is called a **Euclidean quiver** if the underlying graph $\overline{Q}$ of Q is any of the Euclidean diagrams $\widetilde{\mathbb{A}}_m$, with $m \geq 1$, $\widetilde{\mathbb{D}}_m$, with $m \geq 4$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, and $\widetilde{\mathbb{E}}_8$. Analogously, Q is called a **Dynkin quiver** if the underlying graph $\overline{Q}$ of Q is any of the Dynkin diagrams $\mathbb{A}_m$, with $m \geq 1$, $\mathbb{D}_m$, with $m \geq 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$.

We take pleasure in thanking all our colleagues and students who helped us with their comments and suggestions. We wish particularly to express our appreciation to Ibrahim Assem, Sheila Brenner, Otto Kerner, and Kunio Yamagata for their helpful discussions and suggestions. Particular thanks are due to Dr. Jerzy Białkowski and Dr. Rafał Bocian for their help in preparing a print-ready copy of the manuscript.

Contents

Chapter X

Tubes

In Chapter VIII of Volume 1, we have started to study the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of any hereditary K-algebra A of Euclidean type, that is, the path algebra $A = KQ$ of an acyclic quiver Q whose underlying graph $\overline{Q}$ is one of the Euclidean diagrams $\widetilde{\mathbb{A}}_m$, with $m \geq 1$, $\widetilde{\mathbb{D}}_m$, with $m \geq 4$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, and $\widetilde{\mathbb{E}}_8$. We recall that any such an algebra A is representation-infinite.

We have shown in (VIII.2.3) that the quiver $\Gamma(\operatorname{mod} A)$ contains a unique postprojective component $\mathcal{P}(A)$ containing all the indecomposable projective A-modules, a unique preinjective component $\mathcal{Q}(A)$ containing all the indecomposable injective A-modules, and the family $\mathcal{R}(A)$ of the remaining components being called regular (see (VIII.2.12)). This means that $\Gamma(\operatorname{mod} A)$ has the disjoint union form

$$\Gamma(\operatorname{mod} A) = \mathcal{P}(A) \cup \mathcal{R}(A) \cup \mathcal{Q}(A).$$

The indecomposable modules in $\mathcal{R}(A)$ are called regular. We have shown in (VIII.4.5) that there is a similar structure of $\Gamma(\operatorname{mod} B)$, for any concealed algebra B of Euclidean type, that is, the endomorphism algebra

$$B = \operatorname{End} T_A$$

of a postprojective tilting module T_A over a hereditary algebra $A = KQ$ of Euclidean type. The algebra B is representation-infinite.

The objective of Chapters XI-XIII is to describe the structure of regular components of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} B)$ of any concealed algebra B of Euclidean type.

We introduce in this chapter a special type of a translation quiver, which we call a stable tube. The main aim of Section 1 is to describe special properties of irreducible morphisms between indecomposable modules in stable tubes of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} B)$ of an algebra B and their compositions with arbitrary homomorphisms in the module category $\operatorname{mod} B$. In particular, some relevant properties of the radical rad_B and the infinite radical $\operatorname{rad}_B^\infty$ of the category $\operatorname{mod} B$ of finite dimensional right B-modules are described.

In Section 2, we introduce the important concept of a standard component and we prove Ringel's handy criterion on the existence of a standard self-hereditary stable tube in the Auslander–Reiten quiver $\Gamma(\operatorname{mod} B)$ of any algebra B. By applying the criterion, we show in Chapter XI that the regular components of any (representation-infinite) concealed algebra B of Euclidean type are self-hereditary standard stable tubes.

In Section 3, we introduce the concept of a generalised standard component of $\Gamma(\text{mod}\, B)$, invoking the infinite radical rad_B^∞ of the category $\text{mod}\, B$, and exhibit basic examples of generalised standard components. The main result of Section 4 is a characterisation of (generalised) standard stable tubes obtained by Skowroński in [246], [247], and [254]. It asserts that, for a stable tube $\mathcal{T}$ in the Auslander–Reiten quiver $\Gamma(\text{mod}\, B)$ of any algebra B, the following three statements are equivalent:

- $\mathcal{T}$ is a standard stable tube,
- the mouth of $\mathcal{T}$ consists of pairwise orthogonal bricks, and
- $\mathcal{T}$ is a generalised standard stable tube.

It is also shown that $\text{pd}\, X = 1$ and $\text{id}\, X = 1$, for any indecomposable B-module lying in a faithful generalised standard stable tube $\mathcal{T}$ of $\Gamma(\text{mod}\, B)$.

Throughout, we assume that K is an algebraically closed field, and by an algebra we mean a finite dimensional K-algebra. Given a finite quiver $Q = (Q_0, Q_1)$, we denote by KQ the path K-algebra of Q. We recall that the dimension $\dim_K KQ$ of KQ is finite if and only if the quiver Q is **acyclic**, that is, there is no oriented cycle in Q, see Chapters II and III.

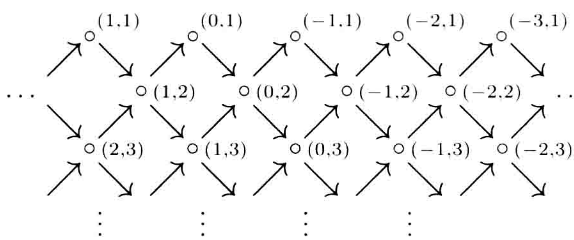
X.1. Stable tubes

We have defined in (VIII.1.1) the translation quiver $\mathbb{Z}\Sigma$, for Σ being a connected and acyclic quiver. Thus, letting Σ be the infinite quiver

$$\mathbb{A}_\infty : \quad \underset{1}{\circ} \longrightarrow \underset{2}{\circ} \longrightarrow \underset{3}{\circ} \longrightarrow \underset{4}{\circ} \longrightarrow \cdots \longrightarrow \underset{m}{\circ} \longrightarrow \underset{m+1}{\circ} \longrightarrow \cdots$$

we obtain the **infinite translation quiver**

$\mathbb{Z}\mathbb{A}_\infty$:



where $\tau(n, i) = (n + 1, i)$, for $n \in \mathbb{Z}$ and $i \geq 1$. Thus, by definition, τ is an automorphism of $\mathbb{Z}\mathbb{A}_\infty$, and hence so is any power τ^r of τ (with $r \in \mathbb{Z}$). For a fixed $r \geq 1$, let (τ^r) denote the infinite cyclic group of automorphisms of $\mathbb{Z}\mathbb{A}_\infty$ generated by τ^r, and let $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ denote the orbit space of $\mathbb{Z}\mathbb{A}_\infty$ under the action of (τ^r). That is, $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is the translation quiver obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each point (n, i) of $\mathbb{Z}\mathbb{A}_\infty$ with the point $\tau^r(n, i) = (n + r, i)$, and each arrow $\alpha : x \to y$ in $\mathbb{Z}\mathbb{A}_\infty$ with the arrow $\tau^r\alpha : \tau^r x \to \tau^r y$. We are thus led to the following definition.

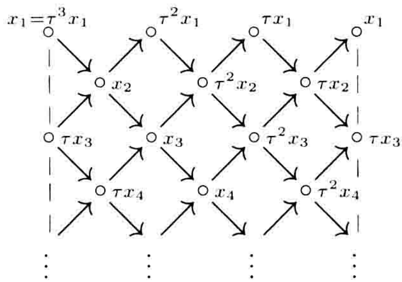
1.1. Definition. Let $(\mathcal{T}, \tau)$ be a translation quiver.

(a) $(\mathcal{T}, \tau)$ is defined to be a **stable tube of rank** $r = r_{\mathcal{T}} \geq 1$ if there is an isomorphism of translation quivers $\mathcal{T} \cong \mathbb{Z}\mathbb{A}_\infty/(\tau^r)$.

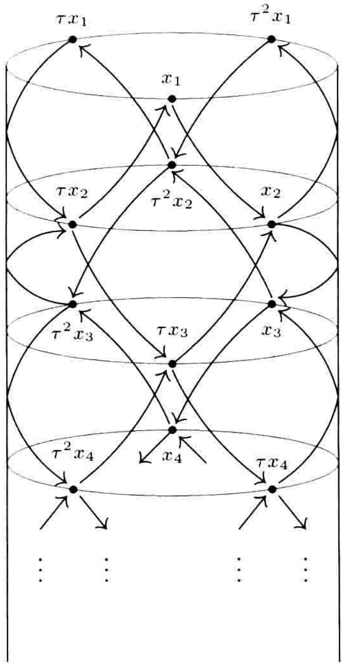
(b) A stable tube of rank $r = 1$ is defined to be a **homogeneous tube**.

(c) Let $(\mathcal{T}, \tau)$ be a stable tube of rank $r \geq 1$. A sequence $(x_1, \ldots, x_r)$ of points of $\mathcal{T}$ is said to be a **τ-cycle** if $\tau x_1 = x_r, \tau x_2 = x_1, \ldots, \tau x_r = x_{r-1}$.

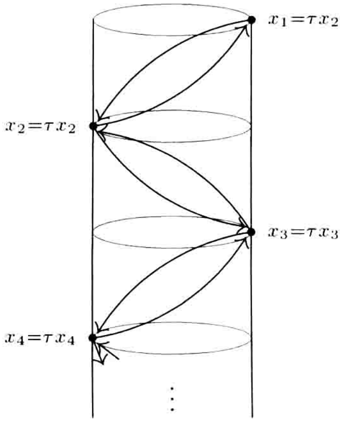
For example, a stable tube of rank 3 is obtained from the quiver



by identifying along the vertical dotted lines, thus giving the following



Similarly, a homogeneous tube has the following shape



We observe that the translation τ still acts as an automorphism over a stable tube of rank r (that is the reason why such tubes are called stable), and that τ^r acts as the identity. The latter fact is expressed by saying that any point of $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ is **τ-periodic** of period r.

We recall from Section IX.2 that a path $x_0 \to \cdots \to x_t$ in a translation quiver is called **sectional** if $\tau x_i \not\cong x_{i-2}$, for all $i \in \{2, \ldots, t\}$.

The following two definitions are of importance in the theory.

1.2. Definition. Let $(\mathcal{T}, \tau)$ be a stable tube.

(a) The set of all points in $\mathcal{T}$ having exactly one immediate predecessor (or, equivalently, exactly one immediate successor) is called the **mouth** of $\mathcal{T}$.

(b) Given a point x lying on the mouth of the stable tube $\mathcal{T}$, a **ray** starting at x is defined to be a unique infinite sectional path

$$x = x[1] \longrightarrow x[2] \longrightarrow x[3] \longrightarrow x[4] \longrightarrow \ldots \longrightarrow x[m] \longrightarrow \ldots$$

in the tube $\mathcal{T}$.

(c) Given a point x lying on the mouth of the stable tube $\mathcal{T}$, a **coray** ending with x is defined to be a unique infinite sectional path

$$\ldots \longrightarrow [m]x \longrightarrow \ldots \longrightarrow [4]x \longrightarrow [3]x \longrightarrow [2]x \longrightarrow [1]x = x$$

in the tube $\mathcal{T}$.

To see that the definition is correct, we note that, for each point x lying on the mouth of a stable tube $\mathcal{T}$, there exists a unique arrow starting at x and a unique arrow ending at x. Because an arbitrary point in $\mathcal{T}$ is the source (and the target) of at most two arrows, this implies the existence of a unique infinite sectional path in $\mathcal{T}$ starting at x, and a unique infinite sectional path in $\mathcal{T}$ ending with x.

1.3. Definition. Let A be an algebra and $\mathcal{C}$ be a component of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of A.

(a) A **ray point** of $\mathcal{C}$ is defined to be a point X in $\mathcal{C}$ such that there exists an infinite sectional path in $\mathcal{C}$

$$X = X[1] \longrightarrow X[2] \longrightarrow X[3] \longrightarrow X[4] \longrightarrow \dots \longrightarrow X[m] \longrightarrow \dots$$

starting at X and containing all sectional paths starting at X. The corresponding A-module X is called a **ray module**. The unique infinite sectional path starting at X is called the **ray** starting at X.

(b) A **coray point** of $\mathcal{C}$ is defined to be a point X in $\mathcal{C}$ such that there exists an infinite sectional path in $\mathcal{C}$

$$\dots \longrightarrow [m]X \longrightarrow \dots \longrightarrow [4]X \longrightarrow [3]X \longrightarrow [2]X \longrightarrow [1]X = X$$

ending with X and containing all sectional paths ending with X. The corresponding A-module X is called the **coray module**. The unique infinite sectional path ending with X is called the **coray** ending with X.

A **ray point of a stable tube** $\mathcal{T}$ and a **coray point of a stable tube** $\mathcal{T}$ are defined analogously.

It is easy to see that if $(\mathcal{T}, \tau)$ is a stable tube and x is a point x in $\mathcal{T}$, then the following three statements are equivalent:

- x is a ray point of the tube $\mathcal{T}$,
- x is a coray point of $\mathcal{T}$, and
- x lies on the mouth of the tube $\mathcal{T}$.

Now we collect basic facts on the structure of any stable tube of $\Gamma(\operatorname{mod} A)$.

1.4. Lemma. *Let A be an algebra, and $\mathcal{T}$ a stable tube of rank $r = r_{\mathcal{T}} \geq 1$ of the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of A. Assume that $(X_1, \dots, X_r)$ is a τ_A-**cycle** of (indecomposable) mouth modules of mouth A-modules of the tube $\mathcal{T}$, that is, the modules $X_1, \dots, X_r$ lie on the mouth of $\mathcal{T}$ and satisfy $\tau_A X_1 \cong X_r, \tau_A X_2 = X_1, \dots, \tau_A X_r = X_{r-1}$.*

(a) *For each $i \in \{1, \dots, r\}$, there exists a unique ray*

$$(\mathfrak{r}_i)\quad X_i = X_i[1] \longrightarrow X_i[2] \longrightarrow X_i[3] \longrightarrow \dots \longrightarrow X_i[m] \longrightarrow X_i[m+1] \longrightarrow \dots$$

in $\mathcal{T}$ starting at X_i, and a unique coray

$$(\mathfrak{c}_i)\quad \dots \longrightarrow [m+1]X_i \longrightarrow [m]X_i \longrightarrow \dots \longrightarrow [3]X_i \longrightarrow [2]X_i \longrightarrow [1]X_i = X_i$$

in $\mathcal{T}$ ending with X_i.

(b) *Every indecomposable A-module M in $\mathcal{T}$ is of the form $M \cong X_i[m]$, for some $i \in \{1, \dots, r\}$ and $m \geq 1$.*

(c) *Every indecomposable A-module M in $\mathcal{T}$ is of the form $M \cong [m]X_s$, for some $s \in \{1, \dots, r\}$ and $m \geq 1$.*

(d) $[m]X_s \cong X_{s-m+1}[m]$ *and* $X_s[m] \cong [m]X_{s+m-1}$, *for each* $s \in \{1, \dots, r\}$ *and* $m \geq 1$, *where* $s - m + 1$ *is reduced modulo* $r - 1$ *if* $s - m + 1 \leq 0$ *or* $s - m + 1 \geq r$.

(e) *Under the isomorphisms of A-modules*

$$[1]X_i \cong X_i[1],\ [2]X_i \cong X_{i-1}[2],\ \dots,\ [m]X_i \cong X_{i+m-1}[m], \dots$$

the coray $(\mathfrak{c}_i)$ has the form

$$(\mathfrak{c}_i)\quad \dots \to X_{i-m}[m+1] \longrightarrow X_{i-m+1}[m] \longrightarrow \dots \longrightarrow X_{i-1}[2] \longrightarrow X_i[1] = X_i.$$

(f) *For any $i \in \{1, \dots, r\}$ and $m \geq 1$, there exists an almost split sequence*

$$0 \longrightarrow X_i[m] \xrightarrow{\begin{bmatrix} f_{i,m+1} \\ g_{i,m} \end{bmatrix}} X_i[m+1] \oplus X_{i+1}[m-1] \xrightarrow{[g_{i,m+1}\ f_{i+1,m}]} X_{i+1}[m] \longrightarrow 0$$

in $\operatorname{mod} A$, *where we set* $X_i[0] = 0$ *and* $X_{i+kr}[j] = X_i[j]$, *for all* $i \in \{1, \dots, r\}$, $j \geq 1$, *and* $k \in \mathbb{Z}$.

Proof. Assume that $(X_1, \dots, X_r)$ is a τ_A-cycle of mouth modules of the tube $\mathcal{T}$ of rank $r \geq 1$. Then $X_1, \dots, X_r$ are indecomposable, lie on the mouth of the tube $\mathcal{T}$, and there are isomorphisms $\tau_A X_1 \cong X_r, \tau_A X_2 = X_1, \dots, \tau_A X_r = X_{r-1}$. Because the tube $\mathcal{T}$ is stable then there is a surjective morphism $f : \mathbb{Z}\mathbb{A}_\infty \longrightarrow \mathcal{T}$ of translation quivers such that $f(-1, 1) = X_1, f(-2, 1) = X_2, \dots, f(-r, 1) = X_r$ and the induced morphism $\widetilde{f} : \mathbb{Z}\mathbb{A}_\infty/(\tau^r) \xrightarrow{\simeq} \mathcal{T}$ is an isomorphism. It is clear that the conditions (a), (b), and (c) are satisfied in the translation quiver $(\mathbb{Z}\mathbb{A}_\infty, \tau)$. Hence we easily conclude that (a), (b), and (c) hold in $\mathcal{T}$, if we set $X_i[m] = f(-i, m)$ and $[m]X_i = f(-i+m-1, m)$.

The statement (d) follows from (a), (b), and (c) by an easy induction on $m \geq 1$.

Now we prove (e). It follows from (d) that, given $i \in \{1, \dots, r\}$ and $m \geq 1$, there are isomorphisms $X_{i-m}[m+1] \cong [m+1]X_i$ and $X_{i-m+1}[m] \cong [m]X_i$. Hence, the following arrow in the coray $(\mathfrak{c}_i)$

$$X_{i-m}[m+1] \cong [m+1]X_i \longrightarrow [m]X_i \cong X_{i-m+1}[m]$$

corresponds to an irreducible morphism $X_{i-m}[m+1] \longrightarrow X_{i-m+1}[m]$ in $\operatorname{mod} A$. To prove (f), we note that in view of the shape of the stable tube $\mathcal{T}$, each of its vertices is a source of at most two arrows, and the arrows correspond to some irreducible morphisms in $\operatorname{mod} A$; thus yield a required almost split sequence. The proof of the lemma is then complete. □

In the remaining part of this section we investigate properties of irreducible morphisms between indecomposable modules in a stable tube $\mathcal{T}$ of