Elastic Instability Phenomena

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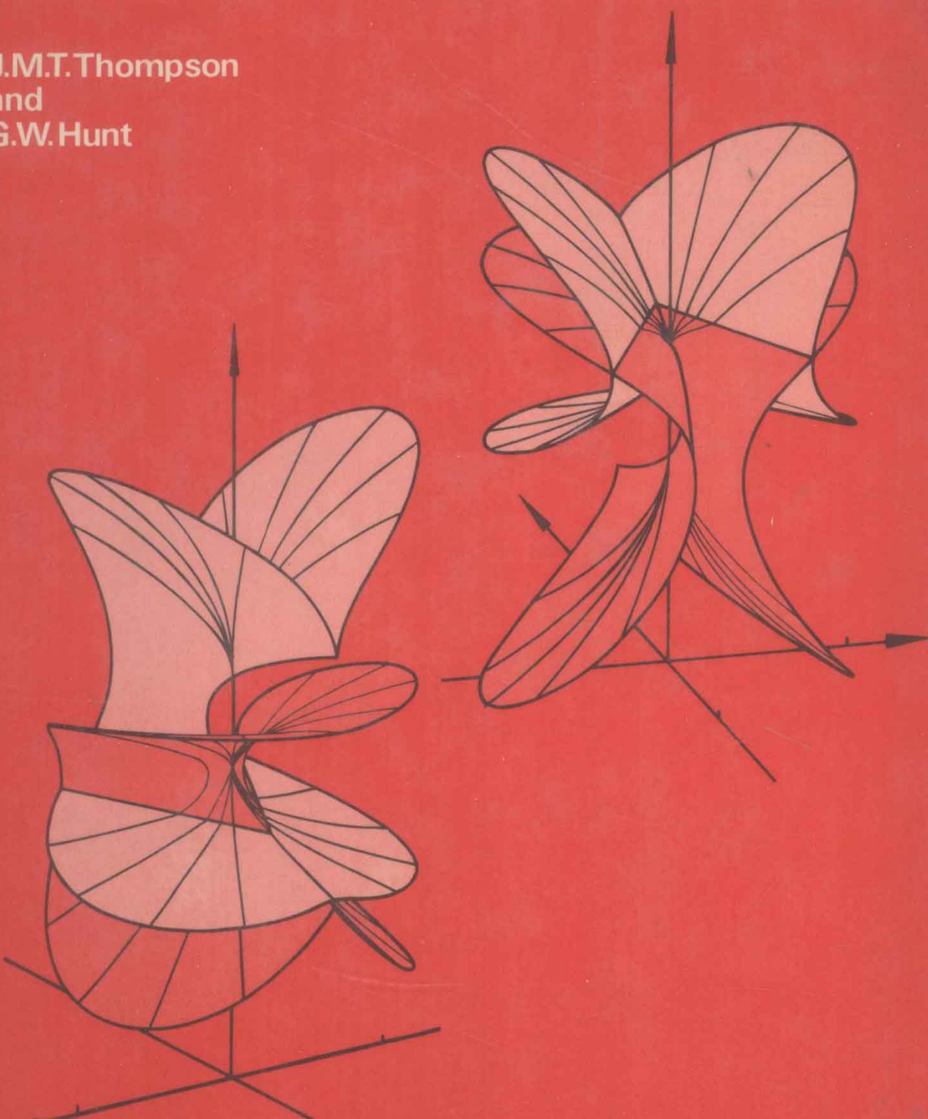

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Preface

The present work treats the buckling and post-buckling of engineering structures and components, with an emphasis on the important non-linear features of behaviour. The subject is developed from the foundation of structural dynamics, and concentrates on the buckling of structures under conservative loading, including the significant imperfection-sensitivity of stiffened plates and shells.

The book is intended for undergraduate and post-graduate courses in civil, mechanical, marine, and aerospace engineering, and includes material that has been taught for a number of years to undergraduate and MSc students of Civil Engineering at University College and Imperial College, London. It can be seen as a complement to two earlier books by the same authors. The first, *A General Theory of Elastic Stability*,[1] by J. M. T. Thompson and G. W. Hunt, published by John Wiley in 1973, was an advanced text on non-linear bifurcation theory, dealing with the elastic buckling and post-buckling of conservative mechanical systems and structures. The second, *Instabilities and Catastrophes in Science and Engineering*,[2] by J. M. T. Thompson, published by John Wiley in 1982, described in a more general, less technical way a wide variety of instability phenomena drawn from the breadth of science and technology. It dealt with the dynamic flutter instabilities of non-conservative systems in addition to the static buckling instabilities of conservative systems, and related both to recent developments in dynamical systems theory, of which catastrophe theory is a part.

This book returns to the theme and spirit of the first of those works,[1] which it complements and completes in a number of important ways. It is restricted to conservative systems, and presents *first*, the underlying *dynamical framework* in which instability problems should properly be viewed: these foundations, based on the Lagrange equations, were not presented explicitly by Thompson and Hunt.[1] *Second*, it relates the engineering theory of elastic stability to recent advances in *singularity theory* which come under the name of catastrophe theory. This broadens the context of the work, and adds some useful new points of view, including the important mathematical concept of a structurally stable topology. A significant *third* contribution is a full extension of the earlier outline of *interactive buckling* at compound branching points, in which the topological view is found to contribute non-trivially, leading to a deeper understanding than had hitherto been available.

Closely related as it is to our earlier books, we have nevertheless tried to make this book as self-contained as possible. This has inevitably involved a little repetition and overlapping of our previous work, but we have kept this to a minimum by presenting, whenever possible, an alternative view using different illustrative examples.

Chapter 1 lays the conceptual foundations of stability theory. The principles are developed gradually but systematically through physically understandable examples, starting with the simple oscillations of a pendulum. It gives a thorough treatment of the general conservative mechanical system based on the Lagrange equations. Stability is defined in the manner of Liapunov by the form of the phase trajectories close to an equilibrium state, and the powerful energy theorems are established. Linear and non-linear conditions for a minimum of the total potential energy are presented, leading to the definition of stability coefficients and normal modes of vibration.

Chapter 2 presents multi-mode linear eigenvalue analyses of beams and columns. Fourier expansions yield complete closed-form results for the vibration and buckling of struts, while discontinuous Rayleigh–Ritz modal analyses serve as an introduction to finite element methods.

Chapter 3 introduces loads and imperfections as control parameters modifying the potential energy of the system. It looks briefly at related fields that are covered by the present theory,[2] including the thermodynamics of stars and the fracture of crystals. The elimination of passive coordinates is presented, showing how the buckling modes alone govern the incipient instability of a structure. Evolution under an increasing load leads to a bifurcational view of instability phenomena, and four common distinct critical points are delineated, along with two basic theorems recently proved by mathematical topologists. Imperfections perturb these bifurcations, generating imperfection-sensitivity phenomena. Mathematical ideas of topological stability allow the numeration and identification of active control parameters (loads, imperfections, and geometrical parameters) essential for the full description of a given singularity. The chapter ends by looking at the *catastrophe theory* classifications of Thom and Zeeman[3–5] and the related but finer *bifurcational classifications* of Golubitsky and Schaeffer.[6]

Chapter 4 looks in detail at single-mode buckling phenomena at distinct critical points. For each form of instability the simplest universal expression for the local governing potential function is given: link models, struts, and frames are used as illustrative examples. The fold catastrophe is seen to generate first, the limit or snap-buckling point, and second, within a bifurcational view, the asymmetric point of bifurcation. The cusp generates the stable-symmetric and unstable-symmetric points of bifurcation: requirements of structural stability under general and bifurcational formalisms are discussed. Routes through catastrophes are used to illustrate the emergence of the bifurcational viewpoint, and a non-symmetric control route through a cusp is seen to generate a cut-off point familiar to engineers in the buckling of shallow arches and domes. Two higher-order one-mode singularities are finally discussed, with a consideration of the reduced Euler buckling load used in approximate engineering analysis.

Chapter 5 is devoted to a new analysis of stress-free and pre-stressed arches which are now used almost universally as a classical laboratory demonstration of imperfection-sensitivity in the unstable-symmetric point of bifurcation. Buckling, post-buckling, and imperfection-sensitivity predictions agree well with the available experimental results of Roorda. The explicit identification of a constraint condition throws new light on the complex contorted equilibrium paths of arches under central point load.

Chapter 6 starts by looking at the simultaneous compound buckling of a spatially guyed cantilever. The post-buckling and imperfection-sensitivity of the semi-symmetric points of bifurcation are then discussed, including the effects of four control parameters, namely the load, an imperfection in each buckling mode, and a splitting parameter (normally a geometric parameter in a particular buckling problem) that controls the gap between the two distinct bifurcations involved. As an example of the universal unfolding of the umbilic catastrophes, the homeoclinal, anticlinal, and monoclinal points of bifurcation are studied in detail, and computed imperfection-sensitivity surfaces are presented and described. Fully asymmetric points of bifurcation are classified using a Lagrange Multiplier technique, and routes through the umbilic catastrophes are explored. Higher-order two-mode singularities are sketched, including parabolic umbilic, recently identified as of fundamental significance in interactive buckling, and the double cusp, which is relevant to the compound buckling of elastic plates.

Chapter 7 presents a comprehensive bifurcation analysis of the general system governed by a potential energy function of n coordinates and h controls, first published in the *Philosophical Transactions of the Royal Society*. The system is presumed to exhibit a single-valued fundamental path under the evolution of a distinctive primary control, normally a load. Sliding coordinates are then used to define a new incremental energy function. Active and passive coordinates are segregated, and the latter are eliminated locally using a preliminary perturbation scheme to give a transformed energy function of the active coordinates alone: the original equilibrium and stability axioms hold good for this new energy function of m active coordinates. Appropriate identities for the specification of equilibrium states, critical states, and secondary bifurcations are presented explicitly, and alternative perturbation schemes are outlined. The concepts of generalized loads and generalized imperfections are introduced, and some hints for computer solutions are given. The comprehensive bifurcation analysis is illustrated by application to the semi-symmetric branching points which involve compound buckling in two simultaneous modes.

The last chapter looks at some engineering buckling problems in the light of the foregoing phenomena and analytical techniques. The elimination of passive coordinates within a diagonalized energy formulation shows explicitly how the *total* quartic energy derivatives with respect to the active coordinates are contaminated by cubic derivatives of the non-critical passive buckling modes: this is a crucial point in the following discussions. A brief look at structural optimization and its associated symmetries highlights the important role that these play in all buckling problems.

The distinctive post-buckling of struts and plates are then examined, and an approximate energy analysis due to Koiter is used for the latter to highlight the previously mentioned general features. The practically important interactive buckling of stiffened structures, very much a current research topic,[7] is examined in the light of the controlling parabolic umbilic catastrophe. The higher-order singularities governing the instability of compressed cylindrical and spherical shells are discussed following Koiter's classic contributions. The chapter ends with a brief discussion of the dead and rigid loading of laboratory model structures, while a proof of the Lagrange and Hamilton equations is given as an Appendix.

An extensive list of modern references supplements and updates the comprehensive lists in our earlier books.[1,2,7]

Contents

1

The General Conservative System

In this opening chapter we consider the dynamics and stability of a general n-degree-of-freedom conservative mechanical system, based on the Lagrange equations of motion. Here we define a conservative mechanical system as one whose generalized forces are completely derivable from a potential energy function $V(Q_i)$. We thus exclude gyroscopic systems, despite the fact that they do also conserve energy. We admit, however, on occasions the presence of a small amount of positive definite viscous damping. This changes pathological centres into asymptotically-stable foci in the phase-space, and allows the proof of the converse of the Lagrange energy theorem.

The Liapunov definition of stability is employed, and the equilibrium and stability axioms necessary for our later work are established. Conditions for a minimum of the total potential energy are analysed at length, and the chapter ends with a presentation of normal-mode linear vibration theory.

Discrete mechanical models are analysed throughout the chapter, and the theory is further illustrated by the strut analyses of the following chapter.

1.1 GENERALIZED COORDINATES

Spatial configurations of our general mechanical system are to be specified by a set of n *generalized coordinates*, written as

$$Q_1, Q_2, Q_3, \dots Q_n$$

or more briefly as Q_i, where i is understood to take values from one to n. Here the number n represents the *degree of freedom* of the system. We require that there shall be a unique one-to-one correspondence between the spatial configurations of our system and the set of values of the coordinates. Thus if we introduce an n-dimensional *state space* by associating the algebraic variables Q_i with a set of n rectangular axes, as shown schematically in Figure 1.1, there will be a unique, one-to-one correspondence between spatial states of the system and the points of this space.

This one-to-one correspondence need not always be global, but must hold locally to the region of our immediate discussion. For example, local configurations of the rigid pendulum of Figure 1.2 can always be specified by the

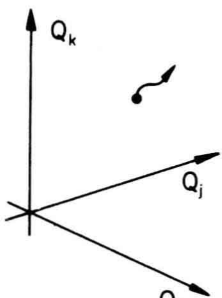

Figure 1.1 Schematic diagram of a trajectory in state space

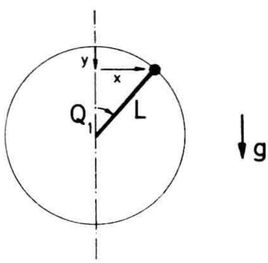

Figure 1.2 A simple rigid pendulum

angle Q_1 as a single generalized coordinate. Alternatively, the horizontal displacement x can be employed, being valid except in the region of $Q_1 =$ 90 degrees or 270 degrees: as a second alternative generalized coordinate we can employ the vertical displacement y except in the region of $Q_1 = 0$ degrees or 180 degrees.

For the small-but-finite deflections of a beam or stretched wire, the mathematical continuum problem can be *discretized* by either a classical modal analysis or by a numerical finite-element analysis. In the classical analysis we might, for example, use the harmonic amplitudes Q_1, Q_2, Q_3,... as our generalized coordinates as illustrated in Figure 1.3, and the fact that the number of coordinates, n, is now strictly infinite will be largely ignored in our discussions. This step, although naturally distressing to mathematicians, rarely gives rise to any real problems in physics and engineering.

Just as any point in state space represents a unique admissible spatial position of the system, so a trajectory in state space is assumed to represent a unique admissible motion of the system. In the terminology of Synge and Griffith[8] the general mechanical system is therefore *holonomic* with, for example, no differential constraints that can arise in the three-dimensional rolling of wheels.

The general system is also assumed to have no time-dependent constraints and

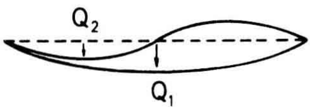

Figure 1.3 Discretization of a continuum by the use of Fourier harmonics

no time-dependent forces, so the time t will not appear explicitly in the potential or kinetic energy expressions, and it is therefore also *scleronomic*.[8]

1.2 LAGRANGE EQUATIONS

We now suppose our general mechanical system to be *conservative* and *undamped*, comprising, for example, internal elastic elements and external conservative fields (gravitational or otherwise), so that all forces of the system are derivable from a *total potential energy function* V which is a function of only the n generalized coordinates Q_i. We write this simply as

$$V = V(Q_i) \tag{1.1}$$

The kinetic energy T will be primarily a quadratic function of the rates of change of the coordinates

$$\frac{\mathrm{d}Q_i}{\mathrm{d}t} \equiv \dot{Q}_i \tag{1.2}$$

so we can write

$$\begin{aligned} T = {} & \tfrac{1}{2}T_{11}\dot{Q}_1^2 + \tfrac{1}{2}T_{12}\dot{Q}_1\dot{Q}_2 + \cdots + \tfrac{1}{2}T_{1n}\dot{Q}_1\dot{Q}_n \\ & + \tfrac{1}{2}T_{21}\dot{Q}_2\dot{Q}_1 + \tfrac{1}{2}T_{22}\dot{Q}_2^2 + \cdots + \tfrac{1}{2}T_{2n}\dot{Q}_2\dot{Q}_n \\ & + \tfrac{1}{2}T_{31}\dot{Q}_3\dot{Q}_1 + \cdots \\ & + \cdots \end{aligned} \tag{1.3}$$

This can be written more compactly as

$$T = \frac{1}{2}\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} T_{ij}\dot{Q}_i\dot{Q}_j \tag{1.4}$$

and even more concisely as

$$T = \tfrac{1}{2}T_{ij}\dot{Q}_i\dot{Q}_j \tag{1.5}$$

if we adopt the dummy-suffix or tensor notation due to Einstein, which says that any suffix occurring more than once in a product must be summed over all its values. In an expression of this type in which the ultimate meaningful coefficient of $\dot{Q}_5\dot{Q}_7$ (say) is $\tfrac{1}{2}(T_{57} + T_{75})$ it is convenient to specify, as we are quite free to, that the matrix T_{ij} is symmetric, so the $T_{57} = T_{75}$, etc.: the ultimate meaningful coefficient can then be written more compactly as simply T_{57}.

This form for the kinetic energy is quite general if we acknowledge that the coefficients T_{ij} may themselves be functions of the generalized coordinates (but never a function of the rates, or of the time t explicitly). Whether, in a given problem, the coefficients T_{ij} are, or are not, functions of the Q_i depends not necessarily just on the system but also on the coordinates used to describe the system, as we shall illustrate shortly for the simple rigid pendulum. To remind us of this possible dependence we write the set of coefficients T_{ij} as $T_{ij}(Q_k)$, so that our final form for the *kinetic energy function* is

$$T = \tfrac{1}{2}T_{ij}(Q_k)\dot{Q}_i\dot{Q}_j \tag{1.6}$$

Since no elemental contribution to the kinetic energy of a system can be negative, this is a positive–definite form being always positive unless the system is completely at rest (when it is zero).

If we introduce the *Lagrangian function*, $\mathscr{L}$, as simply the difference between these two energies,

$$\mathscr{L}(Q_i, \dot{Q}_j) \equiv T(Q_i, \dot{Q}_j) - V(Q_k) \tag{1.7}$$

all dynamical motions of our general system will be governed by the Lagrange equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{Q}_1} - \frac{\partial \mathscr{L}}{\partial Q_1} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{Q}_2} - \frac{\partial \mathscr{L}}{\partial Q_2} &= 0 \\ &\dots \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{Q}_n} - \frac{\partial \mathscr{L}}{\partial Q_n} &= 0 \end{aligned} \tag{1.8}$$

These are written more concisely as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{Q}_i} - \frac{\partial \mathscr{L}}{\partial Q_i} = 0 \quad (i = 1, 2, \dots n) \tag{1.9}$$

where the free suffix i (which is *not* repeated in any product) is understood to yield a set of equations as it takes all its values from one to n (even if this is not specifically indicated).

These Lagrange equations can be established from Newton's Laws, as outlined in the Appendix for a wider class of (non-conservative) systems, or they can themselves be regarded as fundamental.[8]

Because our system is conservative, we can finally note that the *total energy*

$$E = T + V \tag{1.10}$$

will remain constant during any real motion of the system, corresponding to the principle of *conservation of energy*.

Example: a rigid pendulum

We shall illustrate the foregoing theory for the pendulum of Figure 1.2, comprising a light rigid rod of length L, freely pivoted at one end and carrying a concentrated mass m at its other free end. It is assumed to be acted upon by an external gravitational field of uniform strength g, as would be valid if L were small compared with the radius of the earth. This model is of our general type with a single degree of freedom, so that $n = 1$.

The total potential energy of the model, ignoring the mass of the rod, is simply the gravitational potential of the concentrated mass m, and measuring its height arbitrarily from the pivot as datum, we have

$$V(Q_1) = mgL\cos Q_1 \tag{1.11}$$

This can if necessary be expanded as

$$V(Q_1) = mgL(1 - \tfrac{1}{2}Q_1^2 + \cdots) \tag{1.12}$$

valid for small Q_1.

The kinetic energy is simply $\frac{1}{2}m$ times the square of the tip velocity (ignoring again the mass of the rod), so

$$T(\dot{Q}_1) = \tfrac{1}{2}mL^2\dot{Q}_1^2 \tag{1.13}$$

We notice in passing that had we employed not Q_1 but the horizontal displacement x as our single generalized coordinate we would have obtained

$$V(x) = \pm mgL[1 - (x/L)^2]^{1/2} \tag{1.14}$$

and

$$T(x, \dot{x}) = \tfrac{1}{2}mL^2(\dot{x}/L)^2[1 - (x/L)^2]^{-1} \tag{1.15}$$

where we see that T is a function of both $\dot{x}$ and x. This is a perfectly valid formulation (away from the configurations given by $Q_1 = 90$ degrees or 270 degrees) which could be used to obtain the equations of motion in terms of the coordinate x.

We return, however, to our formulation in terms of the angle Q_1, and observing that $\mathscr{L} = T - V$ and

$$\frac{\partial \mathscr{L}}{\partial \dot{Q}_1} = \frac{\partial T}{\partial \dot{Q}_1} = mL^2\dot{Q}_1 \tag{1.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{Q}_1} = mL^2\ddot{Q}_1 \tag{1.17}$$

$$\frac{\partial \mathscr{L}}{\partial Q_1} = -\frac{\partial V}{\partial Q_1} = mgL\sin Q_1 \tag{1.18}$$

the single Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{Q}_1} - \frac{\partial \mathscr{L}}{\partial Q_1} = 0 \tag{1.19}$$

gives

$$mL^2\ddot{Q}_1 - mgL\sin Q_1 = 0 \tag{1.20}$$

This is the exact differential equation of motion that we could have obtained by the straightforward application of Newton's Laws.

1.3 STATICAL EQUILIBRIUM

It follows from the Lagrange equations that the *necessary and sufficient* condition for statical equilibrium involving no motion of the system is the vanishing of all first derivatives of V,

$$V_i \equiv \frac{\partial V}{\partial Q_i} = 0 \quad \text{(for all } i\text{)} \tag{1.21}$$

Here, and subsequently, we are using the 'equals in all respects' symbol $\equiv$ to indicate a simple notational equivalence or definition.

Following our earlier studies[1] we write this condition formally as an axiom, to emphasize its fundamental role in the subsequent theory.

AXIOM I

A stationary value of the total potential energy with respect to the generalized coordinates is necessary and sufficient for the equilibrium of the system.

We shall see that we need only one further axiom based on the total potential energy, concerning the stability of equilibrium, to provide the foundations of a substantial body of practically important work in the theory of elastic stability.[1]

Now a general Taylor or power series expansion of the energy V about an equilibrium state $Q_i = Q_i^E$ can be written in terms of the incremental coordinates

$$q_i \equiv Q_i - Q_i^E \tag{1.22}$$

as

$$V = V^E + \sum_{i=1}^{i=n} \left.\frac{\partial V}{\partial Q_i}\right|^E q_i + \frac{1}{2}\sum_{i=1}^{i=n}\sum_{j=1}^{j=n} \left.\frac{\partial^2 V}{\partial Q_i \partial Q_j}\right|^E q_i q_j$$
$$+ \text{higher-order terms} \tag{1.23}$$

In this we can always ignore the arbitrary constant $V^E \equiv V(Q_i^E)$ and the following linear term vanishes completely by virtue of our equilibrium condition $V_i^E = 0$. The Taylor series thus starts with the quadratic form, and if we write

$$\left.\frac{\partial^2 V}{\partial Q_i \partial Q_j}\right|^E \equiv V_{ij}^E, \text{ etc.} \tag{1.24}$$

and employ the tensor summation convention, we have concisely

$$V = \tfrac{1}{2} V_{ij}^E q_i q_j + \text{higher-order terms} \tag{1.25}$$

Example: a rigid pendulum

Using by way of an illustration our derived total potential energy expression for the rigid pendulum

$$V = mgL\cos Q_1 \tag{1.26}$$

which for small values of Q_1 can be expanded about the inverted equilibrium state $Q_1 = 0$ as

$$V = mgL(1 - \tfrac{1}{2}Q_1^2 + \cdots) \tag{1.27}$$

we have

$$V_1 \equiv \frac{\partial V}{\partial Q_1} = -mgL\sin Q_1 \tag{1.28}$$