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# Contract Theory in Continuous- Time Models



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*To my parents Antun and Vjera  
To my wife Ying and my son Albert*

# Preface

**Why We Wrote This Book** In recent years there has been a significant increase in interest in continuous-time Principal-Agent models and their applications. Even though the approach is technical in nature, it often leads to elegant solutions with clear economic predictions. Our monograph sets out to survey some of the literature in a systematic way, using a general theoretical framework. The framework we find natural and general enough to include most of the existing results is the use of the so-called Stochastic Maximum Principle, in models driven by Brownian Motion. It is basically the Stochastic Calculus of Variations, used to find first order conditions for optimality. This leads to the characterization of optimal contracts through a system of Forward-Backward Stochastic Differential Equations (FBSDE's). Even though there is no general existence theory for the FBSDE's that appear in this context, in a number of special cases they can be solved explicitly, thus leading to the analytic form of optimal contracts, and enabling derivation of many qualitative economic conclusions. When assuming Markovian models, we can also identify sufficient conditions via the standard approach of using Hamilton-Jacobi-Bellman Partial Differential Equations (HJB PDE's).

**Who Is It For** This book is aimed at researchers and graduate students in Economic Theory, Mathematical Economics and Finance, and Mathematics. It provides a general methodological framework, which, hopefully, can be used to develop further advances, both in applications and in theory. It also presents, in its last part, a primer on BSDE's and FBSDE's. We have used the material from the book when teaching PhD courses in contract theory at Caltech and at the University of Zagreb.

**Prerequisites** A solid knowledge of Stochastic Calculus and the theory of SDE's is required, although the reader not interested in the proofs will need more of an intuitive understanding of the related mathematical concepts, than a familiarity with the technical details of the mathematical theory. A knowledge of Microeconomics is also helpful, although nothing more than a basic understanding of utility functions is required.

**Structure of the Book** We have divided the book into an introduction, three main middle parts, and the last part. The introduction describes the three main settings: risk sharing, hidden actions and hidden types. It also presents a simple example of each. Then, each middle part presents a general theory for the three settings, with a variety of special cases and applications. The last part presents the basics of the BSDE's theory and the FBSDE's theory.

**Web Page for This Book** [sites.google.com/site/contracttheorycvitaniczhang/](http://sites.google.com/site/contracttheorycvitaniczhang/). This is a link to the book web page that will be regularly updated with material related to the book, such as corrections of typos.

**Acknowledgements** Our foremost gratitude goes to our families for the understanding and overall support they provided during the times we spent working on our joint research leading to this book, and for the work on the book itself. We are grateful for the support from the staff of Springer, especially Catriona Byrne, Marina Reizakis and Annika Elting. A number of colleagues and students have made useful comments and suggestions, and pointed out errors in the working manuscript, including Jin Ma, Ajay Subramanian, Xuhu Wan, Xunyu Zhou, Hualei Chang and Nikola Sandrić, and anonymous reviewers.

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Of course, we are solely responsible for any remaining errors, and the opinions, findings and conclusions or suggestions in this book do not necessarily reflect anyone's opinions but the authors'.

**Final Word** We hope that you will find the subject of this book interesting in its economic content, and elegant in its mathematical execution. We would be grateful to the careful reader who could inform us of any remaining typos and errors noticed, or any other comments, by sending an e-mail to our current e-mail addresses. Enjoy!

Los Angeles, USA  
April 2012

Jakša Cvitanić  
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# **Part I**

## **Introduction**



# Chapter 1

## Principal–Agent Problem

**Abstract** A Principal–Agent problem is a problem of optimal contracting between two parties, one of which, namely the agent, may be able to influence the value of the outcome process with his actions. What kind of contract is optimal typically depends on whether those actions are observable/contractable or not, and on whether there are characteristics of the agent that are not known to the principal. There are three main types of these problems: (i) the first best case, or risk sharing, in which both parties have the same information; (ii) the second best case, or moral hazard, in which the action of the agent is hidden or not contractable; (iii) the third best case or adverse selection, in which the type of the agent is hidden.

### 1.1 Problem Formulation

The main topic of this volume is mathematical modeling and analysis of contracting between two parties, **Principal** and **Agent**, in an uncertain environment. As a typical example of a **Principal–Agent problem**, henceforth the **PA problem**, we can think of the principal as an investor (or a group of investors), and of the agent as a portfolio manager who manages the investors' money. Another interesting example from Finance is that of a company (as the principal) and its chief executive (as the agent). As may be guessed, the principal offers a contract to the agent who has to perform a certain task on the principal's behalf (in our model, it's only one type of task).

We will sometimes call the principal  $P$  and the agent  $A$ , and we will also call the principal “she” and the agent “he”.

The economic problem is for the principal to construct a contract in such a way that: (i) the agent will accept the contract; this is called an **individual rationality (IR) constraint**, or a **participation constraint**; (ii) the principal will get the most out of the agent's performance, in terms of expected utility. How this should be done in an optimal way, depends crucially on the amount of information that is available to  $P$  and to  $A$ . There are three classical cases studied in the literature, and which we also focus on in this volume: **Risk Sharing (RS)** with symmetric information, **Hidden Action (HA)** and **Hidden Type (HT)**.

**Risk Sharing** The case of Risk Sharing, also called the **first best**, is the case in which  $P$  and  $A$  have the same information. They have to agree how to share the risk between themselves. It is typically assumed that the principal has all the *bargaining power*, in the sense that she offers the contract and also dictates the agent's actions, which the agent has to follow, or otherwise, the principal will penalize him with a severe penalty. Mathematically, the problem becomes a stochastic control problem for a single individual—the principal, who chooses both the contract and the actions, under the IR constraint. Alternatively, it can also be interpreted as a maximization of their joint welfare by a social planner. More precisely, but still in informal notation, if we denote by  $c$  the choice of contract and by  $a$  the choice of action, and by  $U_A$  and  $U_P$  the corresponding utility functions, the problem becomes

$$\max_{c,a} \{E[U_P(c, a)] + \lambda E[U_A(c, a)]\} \quad (1.1)$$

where  $\lambda > 0$  is a Lagrange multiplier for the IR constraint, or a parameter which determines the level of risk sharing. The allocations that are obtained in this way are Pareto optimal.

**Hidden Action** This is the case in which actions of  $A$  are not observable by  $P$ . Because of this, there will typically be a loss in expected utility for  $P$ , and she will only be able to attain the **second best** reward. Many realistic examples do present cases of  $P$  not being able to deduce  $A$ 's actions, either because it may be too costly to monitor  $A$ , or quite impossible. For example, it may be costly to monitor which stocks a portfolio manager picks and how much he invests in each, and it may be quite impossible to deduce how much effort he has put into collecting information for selecting those stocks.

It should be mentioned that the problem is of the same type even if the actions are observed, but cannot be contracted upon—the contract payoff cannot depend directly on  $A$ 's actions.

Due to unobservable or non-contractable actions,  $P$  cannot choose directly the actions she would like  $A$  to perform. Instead, giving a contract  $c$ , she has to be aware which action  $a = a(c)$  will be optimal for the agent to choose. Thus, this becomes a problem of *incentives*, in which  $P$  indirectly influences  $A$  to pick certain actions, by offering an appropriate contract. Because  $A$  can undertake actions that are not in the best interest of the principal, this case also goes under the name of **moral hazard**.

Mathematically, we first have to solve the *agent's problem* for a given fixed contract  $c$ :

$$V_A(c) := \max_a E[U_A(c, a)]. \quad (1.2)$$

Assuming there is one and only one optimal action  $a(c)$  solving this problem, we then have to solve the *principal's problem*:

$$V_P := \max_c \{E[U_P(c, a(c))] + \lambda E[U_A(c, a(c))]\}. \quad (1.3)$$

Problem (1.2) can be very hard given that  $c$  can be chosen in quite an arbitrary way. A standard approach which makes this easier is to assume that the agent does

not control the outcome of the task directly by his actions, but that he chooses the distribution of the outcome by choosing specific actions. More precisely, this will be modeled by having  $A$  choose probability distributions  $P^a$  under which the above expected values will be taken.

**Hidden Type** In many applications it is reasonable to assume that  $P$  does not know some key characteristics of  $A$ . For example, she may not know how capable an executive is, in terms of how much return he can produce per unit of effort. Or,  $P$  may not know what  $A$ 's risk aversion is. Or how rich  $A$  is. An even more fundamental example is of a buyer (agent) and a seller (principal), in which the buyer may be of a type who cares more or cares less about the quality of the product (wine, for example). Those hidden characteristics, or types, may significantly alter  $A$ 's behavior, given a certain contract.

It is typically assumed in the HT case, as we also do in this book, that  $P$  will offer a **menu of contracts**, one for each type, from which  $A$  can choose. Under certain conditions, a so-called **revelation principle** holds, which says that it is sufficient to consider contracts which are **truth-telling**: the agent will reveal his true type by choosing the contract  $c(\theta)$  which was meant for his type  $\theta$ . In particular, the main assumption needed for the revelation principle is that of *full commitment*: once agreed on the contract, the parties cannot change their mind in the future, even if both are willing to renegotiate. This is an assumption that we make throughout.

If the hidden type case is combined with hidden actions, then, generally, the principal gets only her **third best** reward. Since  $A$  can pretend to be of a different type than he really is, which can adversely affect  $P$ 's utility, the hidden type case is also called a case of **adverse selection**. An example is the case of a health insurance company (principal) and an individual (agent) who seeks health insurance, but only if he already has medical problems, and the insurance company may not know about it.

Mathematically, we again first have to solve the agent's problem when he chooses a contract  $c(\theta')$  and he is of type  $\theta$ :

$$V_A(c(\theta'), \theta) := \max_a E^\theta[U_A(c(\theta'), a, \theta)]. \quad (1.4)$$

We assume that the principal's belief about the distribution of types is given by a distribution function  $F(\theta)$ . Denote by  $\mathcal{T}$  the set of truth-telling menus of contracts  $c(\theta)$ . Assuming there is one and only one optimal action  $a(c(\theta'), \theta)$  solving the agent's problem for each pair  $(c(\theta'), \theta)$ , and denoting  $a(c(\theta)) := a(c(\theta), \theta)$  (the action taken when  $A$  reveals the truth) we then have to solve the principal's problem

$$V_P := \max_{c \in \mathcal{T}} \int \{E^\theta[U_P(c(\theta), a(c(\theta)))]) + \lambda(\theta)E^\theta[U_A(c(\theta), a(c(\theta)))])\} dF(\theta). \quad (1.5)$$

Note that the principal faces now an additional, *truth-telling constraint*, that is,  $c \in \mathcal{T}$ , which can be written as

$$\max_{\theta'} V_A(c(\theta'), \theta) = V_A(c(\theta), \theta). \quad (1.6)$$

## 1.2 Further Reading

There are a number of books that have the PA problem as one of the main topics. We mention here Laffont and Martimort (2001), Salanie (2005), and Bolton and Dewatripont (2005), which all contain the general theory in discrete-time, more advanced topics and many applications.

## References

- Bolton, P., Dewatripont, M.: *Contract Theory*. MIT Press, Cambridge (2005)  
Laffont, J.J., Martimort, D.: *The Theory of Incentives: The Principal-Agent Model*. Princeton University Press, Princeton (2001)  
Salanie, B.: *The Economics of Contracts: A Primer*, 2nd edn. MIT Press, Cambridge (2005)

## Chapter 2

# Single-Period Examples

**Abstract** In this chapter we consider simple examples in one-period models, whose continuous versions will be studied later in the book. Principal–Agent problems in single-period models become more tractable if exponential utility functions are assumed. However, even then, there are cases in which tractability requires considering only linear contracts. Optimal contracts which cannot contract upon the agent’s actions are more sensitive to the output than those that can. When the agents’ type is unknown to the principal, the agents of “higher” type may have to be paid more to make them reveal their type.

### 2.1 Risk Sharing

Assume that the contract payment occurs once, at the final time  $T = 1$ , and we denote it  $C_1$ . The principal draws utility from the final value of an *output process*  $X$ , given by

$$X_1 = X_0 + a + B_1 \quad (2.1)$$

where  $B_1$  is a fixed random variable. The constant  $a$  is the action of the agent.

With full information, the principal maximizes the following case of (1.1), with  $g(a)$  denoting a *cost function*:

$$E[U_P(X_1 - C_1) + \lambda U_A(C_1 - g(a))]. \quad (2.2)$$

Setting the derivative with respect to  $C_1$  inside the expectation equal to zero, we get the first order condition

$$\frac{U'_P(X_1 - C_1)}{U'_A(C_1 - g(a))} = \lambda. \quad (2.3)$$

This is the so-called *Borch rule* for risk-sharing, a classical result that says that *the ratio of marginal utilities of P and A is constant at the risk-sharing optimum*.

We assume now that the utility functions are exponential and the cost of action is quadratic:

$$U_A(C_1 - g(a)) = -\frac{1}{\gamma_A} e^{-\gamma_A [C_1 - ka^2/2]}, \quad (2.4)$$