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имени В. А. СТЕКЛОВА
Том 153 (1981)

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by

A. B. Venkov

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АКАДЕМИЯ НАУК

СОЮЗА СОВЕТСКИХ СОЦИАЛИСТИЧЕСКИХ РЕСПУБЛИК

ТРУДЫ

ордена Ленина

МАТЕМАТИЧЕСКОГО ИНСТИТУТА

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CLIII

А. Б. ВЕНКОВ

СПЕКТРАЛЬНАЯ ТЕОРИЯ АВТОМОРФНЫХ ФУНКЦИЙ

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ABSTRACT. This monograph is devoted to the spectral theory of automorphic functions for noncompact Fuchsian groups of the first kind. It treats the following items: a theorem on expansion in eigenfunctions and Selberg's trace formula; the theory of Selberg's zeta-function; spectral asymptotics and the asymptotics of the lengths of the norms of closed geodesics; Roelcke's problem and the theory of Hecke operators; an analogue of the Artin-Takagi theory for Selberg's zeta-function; spectral theory and deformations; the Dirichlet problem on a regular polygon; an arithmetic application. The work is intended for specialists in function theory, spectral theory of operators, and number theory.

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INTRODUCTION

In his seminal paper [51], Atle Selberg introduced fundamental new ideas into the classical theory of automorphic forms, a theory whose origins lie in the works of Riemann, Klein, and Poincaré. These ideas are connected with an extension of the earlier notion of an automorphic function (or form). Instead of an analytic automorphic function, Selberg considered a mapping which is automorphic relative to a given finite-dimensional unitary representation of a discrete group and is an eigenfunction for a commutative ring of elliptic differential operators. At that time Hans Maass' article [36] had appeared, containing similar nonanalytic automorphic "wave" functions defined in a special situation; however, it was Selberg who first took a serious look at Maass' work. In order to implement the new ideas, certain new techniques, not normally used in the classical theory of automorphic functions, were invoked: first, methods from the theory of selfadjoint operators in Hilbert space; then, methods from group representation theory over various fields, methods which turned out to be more natural in spaces of rank greater than one. It was the subsequent global development of Selberg's ideas in the setting of the representation theory of Lie groups which determined the true place of the classical theory of automorphic functions—in both its function theoretic and number theoretic aspects—in the new more general theory, and also clarified the interaction between the old and new theories.

It is now already possible to speak of the "Selberg theory", although this theory is still in its initial stages of development. The foundation of the theory consists of:

1) theorems on expansion in automorphic eigenfunctions of Laplacians;

2) Selberg trace formulas; and

3) the theory of the Selberg zeta-functions.

Here one should also include several very important applications of a theoretical nature (some of which we recently examined in the survey article [66]):

4) applications to global problems in modern number theory, in particular, to the arithmetic theory of automorphic forms (the so-called "Langlands philosophy");

5) ([1]) applications to solving some difficult concrete problems in number theory, for example, the proof of the refined Kummer conjecture on cubic characters (see [17]);

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([1])Further references to articles on these topics in Selberg theory can be found in our survey article [66] and in the works cited there.

6) applications in the theory of geometric and topological invariants of Riemannian manifolds;

7) the Selberg zeta-function from the point of view of analytic number theory, in particular, the Selberg zeta-function and the Riemann zeta-function;

8) Selberg theory and quasiconformal mappings of Riemann surfaces (see [11] and [12]); and

9) the Selberg trace formula from the point of view of the spectral theory of selfadjoint operators in Hilbert space as a model of stationary and nonstationary scattering theories; applications to classical (Dirichlet, Neumann) boundary value problems of mathematical physics.

In this monograph we shall primarily be interested in questions concerning the foundations of Selberg theory and those applications which have a function theoretic character, corresponding in the above list to items 1)–3), 8) and 9), and parts of 6) and 7). We shall call this entire circle of questions the spectral theory of automorphic functions. We are then almost forced to choose to work with weakly symmetric space (in Selberg's sense) and the set of discrete groups acting on it: the Lobachevsky plane and Fuchsian groups of the first kind. In fact, it has recently become apparent that the theory of automorphic functions in spaces of rank greater than one is for objective reasons an arithmetic theory, in essence a branch of modern number theory. Conversely, the hyperbolic plane stands out—even among the other spaces of rank one—by the abundance of different discrete transformation groups, among which the arithmetically defined groups occupy a very modest place; this is what makes the spectral theory of automorphic functions particularly significant. The choice of Fuchsian groups of the first kind is dictated by the good spectral properties of the automorphic Laplacian (finite multiplicity of the continuous spectrum), the Selberg zeta-function (continuation onto the entire plane, a functional equation, low order of meromorphicity), and also by certain traditional applications.

We now give a more detailed description of the contents. Chapter 1 is introductory; it contains the necessary notation and several definitions and auxiliary facts. Chapter 2 is devoted to a proof of the theorem on expansion in eigenfunctions of the automorphic Laplacian $\mathfrak{A}(\Gamma; \chi)$ for an arbitrary Fuchsian group of the first kind Γ and an arbitrary finite-dimensional unitary representation χ of Γ (such a choice of Γ and χ will be written $\Gamma \in \mathfrak{M}$, $\chi \in \mathfrak{R}(\Gamma)$; see §1.2). We shall pay particular attention to the most difficult situation $\Gamma \in \mathfrak{M}_2$, $\chi \in \mathfrak{R}_s(\Gamma)$, where, by definition, $\mathfrak{M}_2$ is the set of all groups $\Gamma \in \mathfrak{M}$ having noncompact fundamental domain, and $\mathfrak{R}_s(\Gamma)$ is the set of all so-called singular representations $\chi \in \mathfrak{R}(\Gamma)$. Whenever $\Gamma \in \mathfrak{M}_2$ and $\chi \in \mathfrak{R}_s(\Gamma)$, the spectrum of the operator $\mathfrak{A}(\Gamma; \chi)$ is not purely discrete; it also contains an absolutely continuous spectrum of multiplicity equal to the total degree of singularity of the representation χ relative to the group Γ.

The proof of the theorem generalizes L. D. Faddeev's proof of a less general theorem on expansion in eigenfunctions of the operator $\mathfrak{A}(\Gamma; 1)$, where $\chi = 1$ is the trivial one-dimensional (and thus singular) representation of $\Gamma \in \mathfrak{M}_2$ (see [9]). The proof is based on a study of the resolvent $\mathfrak{R}(s; \Gamma; \chi)$ of the operator $\mathfrak{A}(\Gamma; \chi)$ using methods from the theory of perturbations of the continuous spectrum of selfadjoint operators. This proof includes several steps. The kernel of the resolvent as an integral operator far from the spectrum is studied in §2.1. We then construct an auxiliary operator $\mathfrak{B}(s; \Gamma; \chi)$, which is uniquely determined by $\mathfrak{R}(s; \Gamma; \chi)$, and, by means of a certain integral equation, continue it analytically to a neighborhood of the spectrum of $\mathfrak{A}(\Gamma; \chi)$ (more precisely, to part of the Riemann surface $\operatorname{Re} s > 0$, which is a two-sheeted covering of the spectral plane containing the spectrum; see §1.4). We shall call this integral equation the *Faddeev equation*, since it was first introduced into the spectral theory of automorphic functions in [9] in the scalar case ($\dim V = 1$, V the space of the representation χ) for the trivial representation χ. The next steps in the proof of the expansion theorem are meromorphic continuation of the kernel of the resolvent to a neighborhood of the spectrum and the investigation of the singular points of the resolvent (§2.2). Finally, by determining the eigenfunctions of the continuous spectrum of $\mathfrak{A}(\Gamma; \chi)$ in terms of its resolvent, finding the scattering matrix and proving certain of its properties, we are able in §2.3 to complete the proof of the theorem on expansion in eigenfunctions of $\mathfrak{A}(\Gamma; \chi)$.

There is another proof of the expansion theorem for $\mathfrak{A}(\Gamma; \chi)$ which is also valid for arbitrary $\Gamma \in \mathfrak{M}_2$ and $\chi \in \mathfrak{R}_s(\Gamma)$. It was published in [46] by Roelcke, who assumed a very essential conjecture concerning meromorphic continuation of Eisenstein series. The conjecture was later proved in a famous paper by Langlands [33]. However, we wish to emphasize that our method of proof is preferable in our case (for the Lobachevsky plane and for any space of rank one), since it enables one to obtain additional information concerning the spectral properties of $\mathfrak{A}(\Gamma; \chi)$, more precisely, the properties of its resolvent; this information is of great importance in constructing an analog of Artin theory for the Selberg zeta-function (see §6.2), and also in studying the spectral properties of $\mathfrak{A}(\Gamma; \chi)$ under a deformation of the group Γ (see §7.1).

Chapter 3 is devoted to a refinement of the theorem on expansion in eigenfunctions of $\mathfrak{A}(\Gamma; \chi)$ relating to the part of the theorem concerning the continuous spectrum. In §3.1 Eisenstein series are defined for a group $\Gamma \in \mathfrak{M}_2$ and a representation $\chi \in \mathfrak{R}_s(\Gamma)$. (These series are the meromorphic continuation of the eigenfunctions of the continuous spectrum of $\mathfrak{A}(\Gamma; \chi)$.) We then construct their Fourier expansion relative to parabolic subgroups $\Gamma_\alpha \subset \Gamma$. Such expansions have only been considered before in the scalar theory ($\dim V = 1$) (see [50] for χ a nontrivial representation, and [28] for χ the trivial representation).* At the end of §3.1 we prove meromorphicity of Eisenstein series (i.e., the basic hypothesis in Roelcke's paper [46]) and meromorphic continuation of the kernel of the resolvent to the entire Riemann surface which is a double covering of the spectral plane. The proof of these results is based on Chapter 2, which, in particular, gives us meromorphicity of the scattering matrix, a functional equation for the scattering matrix, a functional equation for Eisenstein series, and, finally, a functional equation for the kernel of the resolvent of $\mathfrak{A}(\Gamma; \chi)$.

In §3.2 we give a description, based on [46], of the Maass-Selberg relation. The first part of §3.3 is devoted to a certain intrinsic characterization of the subspace of the continuous spectrum, and, to a lesser extent, the subspace of the discrete

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**Translator's note.* The general case was also considered by Polly Moore, *Generalized Eisenstein series: incorporation of a nontrivial representation of* Γ, Ph. D. dissertation, Univ. of Washington, Seattle, Wash., 1979.

spectrum of $\mathfrak{A}(\Gamma; \chi)$ in a suitable Hilbert space $\mathcal{H}(\Gamma; \chi)$. The theory developed here uses elements of the spectral theory of Selberg [52], Roelcke [46], Godement [14], Langlands [33], and Kubota [28], with emphasis on the properties of Eisenstein series and the Maass-Selberg relations, and especially uses the theorem on expansion in eigenfunctions of $\mathfrak{A}(\Gamma; \chi)$, proved in Chapter 2 based on Faddeev's method in the scalar theory of automorphic functions. Part 2) of Theorem 3.3.2 is a version for resolvents of the well-known theorem of Gel'fand and Pjateckiĭ-Šapiro (see [13], Chapter I, §6, or [16], Chapter I, §2). In the second part of §3.3 we introduce a fundamental class of integral operators to be considered in the spectral theory, and we prove some properties of these operators. They were first introduced by Selberg in [51]. The idea of studying these operators by means of the resolvent $\mathfrak{R}(s; \Gamma; \chi)$ arose in the scalar theory in work by L. D. Faddeev, V. L. Kalinin and the author (see [72]).

In §3.4 we derive a vector version of the integral equation, which we call the *Selberg-Neunhoffer equation*. This integral equation was first proposed to Selberg to prove meromorphicity of Eisenstein series in the scalar theory ($\dim V = 1$) (see [50]). Later, it was reconsidered by Neunhoffer in [39], also in the framework of the scalar theory. We note that the study of the Selberg-Neunhoffer equation is the basis for the third of the methods presently known for proving meromorphicity of Eisenstein series for $\Gamma \in \mathfrak{M}_2$ and $\chi \in \mathfrak{R}_s(\Gamma)$ (see [66], §8). Unlike in [39] and [50], in §3.4 we derive the Selberg-Neunhoffer equation ($\dim V \geqslant 1$) from the Faddeev equation, using the information about the resolvent $\mathfrak{R}(s; \Gamma; \chi)$ of $\mathfrak{A}(\Gamma; \chi)$ in Chapter 2, for a single purpose—finding an a priori estimate for the order of meromorphicity of the Eisenstein series and the scattering matrix. In §3.4 we finish our treatment of this theorem.

§3.5 is devoted to studying the properties of the determinant of the scattering matrix. Here we generalize results of the scalar theory due to Selberg (see [50]). The basic result of the section is Theorem 3.5.5, which gives a special canonical product over the zeros and poles of the determinant which is different from the Weierstrass product.

Chapter 4 is concerned with proving the Selberg trace formula in the general situation $\Gamma \in \mathfrak{M}$, $\chi \in \mathfrak{R}(\Gamma)$. Again we emphasize the most difficult and least well-known case $\Gamma \in \mathfrak{M}_2$, $\chi \in \mathfrak{R}_s(\Gamma)$. In §4.1 we prove nuclearity of the operator $K(\Gamma; \chi)\mathfrak{P}_0(\Gamma; \chi)$, where $\mathfrak{P}_0(\Gamma; \chi)$ is the orthogonal projection in the Hilbert space $\mathcal{H}(\Gamma; \chi)$ onto the subspace $\mathcal{H}_0(\Gamma; \chi)$ of cusp-vector-functions. The proof of the theorem is based upon ideas from the theory of perturbations of continuous spectra (see [72]) and results from Chapter 2. In §4.2 we justify the spectral trace formula; this reduces to proving uniform convergence of certain integrals. Our method generalizes the method in [72] and the Selberg-Arthur method for justifying the trace formula for arithmetic groups in the rank one case (see [1]).

In §4.3 we transform the spectral trace formula in §4.2 to the Selberg trace formula. The following special cases of the Selberg trace formula for $\Gamma \in \mathfrak{M}$, and $\chi \in \mathfrak{R}(\Gamma)$ are well known and have often been examined in the literature: 1) $\Gamma \in \mathfrak{M}_1$ and $\chi \in \mathfrak{R}(\Gamma)$ (see [51], [13], [19] and others); 2) $\Gamma \in \mathfrak{M}_2$, $\chi \in \mathfrak{R}_s(\Gamma)$ and $\dim V = 1$ (see [50], [28] and [72]); 3) $\Gamma \in \mathfrak{M}_2$ and $\chi \in \mathfrak{R}_r(\Gamma)$ (see [51]); and 4) Γ is an arithmetic group in $\mathfrak{M}_2$ and $\chi \in \mathfrak{N}_s(\Gamma)$ (see [23], [7], [1] and [18]) (see §1.2 for the notation). However, the general case of the Selberg trace formula for $\Gamma \in \mathfrak{M}_2$ and $\chi \in \mathfrak{N}_s(\Gamma)$ has not been considered before, as far as we know, either in the published literature or in Selberg's lectures at Princeton (1952) and Göttingen (1954) (see [50]); hence, we shall concentrate our attention in §4.3 on this case. The Selberg trace formula which we obtain for a general group $\Gamma \in \mathfrak{M}_2$ and a general representation $\chi \in \mathfrak{N}_s(\Gamma)$ clearly includes all of the earlier trace formulas 1)–4).

At the beginning of §4.4 we give a proof of a vector version of an asymptotic formula which we have referred to as the *Weyl-Selberg formula* (see [68]). In the scalar theory this formula was first obtained by Selberg (see [50]). In our opinion it is the natural generalization, in the spectral theory of a self-adjoint operator whose spectrum is not in general purely discrete, of Hermann Weyl's classical asymptotic formula for the distribution function of the eigenvalues of an operator with purely discrete spectrum. In §4.4 we also give an a priori estimate for the distribution function of the values of the norms of primitive hyperbolic conjugacy classes in a group $\Gamma \in \mathfrak{M}_2$. Using these two results, later in §4.4 we refine the order of meromorphicity of the determinant of the scattering matrix, and, finally, we give an extension of the Selberg trace formula to a broader class of functions than in §4.3.

Chapter 5 is devoted to the theory of the Selberg zeta-function and its spectral and geometric applications in the general situation $\Gamma \in \mathfrak{M}_2$ and $\chi \in \mathfrak{N}(\Gamma)$. In §5.1 we give the definition and prove the basic properties of the Selberg zeta-function $Z(s; \Gamma; \chi)$. The function $Z(s; \Gamma; \chi)$ is connected with the Selberg trace formula in the same way as the Riemann zeta-function is connected with Weil's "explicit formula" in analytic number theory. Thus, all of the basic properties of $Z(s; \Gamma; \chi)$ are determined by the Selberg trace formula. In §5.1 we prove a fundamental formula for the logarithmic derivative of the Selberg zeta-function (Theorem 5.1.1). This formula gives us meromorphicity of $Z(s; \Gamma; \chi)$, a functional equation, and also a complete description of all the zeros and poles of the zeta-function (Theorems 5.1.3 and 5.1.4). In [68] we published similar results for the scalar theory of the zeta-function $Z(s; \Gamma; \chi)(\dim V = 1)$; the stimulus for all of these investigations was the brief remarks of Selberg at the end of his lectures [50].

§5.2 is devoted to estimating the remainder in the Weyl-Selberg asymptotic formula. More precisely, we construct an asymptotic formula with three principal terms and a remainder term of order $O(T/\ln T)$ (Theorem 5.2.1); here the justification for the first principal term is the content of Theorem 4.4.1 in §4.4 and was what lead to the Weyl-Selberg formula. The derivation of the formula is based on the theory of the Selberg zeta-function $Z(s; \Gamma; \chi)$, and is a spectral application of that formula. The method of proof generalized a method well known in analytic number theory for constructing an asymptotic formula for the number of nontrivial zeros of the Riemann zeta-function in a "large" rectangle in the critical strip (see, for example, [57]). Hejhal [19] and Randol [44] obtained the analogous formula for a group $\Gamma \in \mathfrak{M}_1$ and the trivial one-dimensional representation χ, and the author [68] did the same for a group $\Gamma \in \mathfrak{M}_2$ and $\chi \in \mathfrak{N}_s(\Gamma)$ ($\dim V = 1$).

The purpose of §5.3 is to derive an asymptotic formula for the distribution function for the values of the norms of primitive hyperbolic conjugacy classes in a given Fuchsian group. This formula should be regarded as the geometric application of the theory of the Selberg zeta-function $Z(s; \Gamma; \chi)$ which we develop in §§5.1 and 5.2. The formula is analogous to the refined asymptotic law for the distribution of prime natural numbers, and it is connected with the Selberg zeta-function in the same way as that asymptotic law is connected with the Riemann zeta-function. In the theory of the Selberg zeta-function $Z(s; \Gamma; \chi)$ for $\Gamma \in \mathfrak{M}_1$, such a formula is apparently due to Selberg and Huber. There are published proofs in papers by Huber [21] and Hejhal [19]. For our type of group $\Gamma \in \mathfrak{M}_2$ the formula was published by A. I. Vinogradov and the author in the note [71] (see also [66]).

Chapter 6 is largely concerned with a refinement of the theorem on expansion in eigenfunctions of the operator $\mathfrak{A}(\Gamma; \chi)$ for $\Gamma \in \mathfrak{M}_2$ and $\chi \in \mathfrak{R}_s(\Gamma)$ in the aspect relating to the discrete spectrum of $\mathfrak{A}(\Gamma; \chi)$. The first part of §6.1 is an introduction to the chapter as a whole. In this section we formulate the basic problems of the theory of the discrete spectrum that are still unsolved. Special attention is accorded the so-called Roelcke conjecture to the effect that there are infinitely many eigenvalues of the discrete spectrum of $\mathfrak{A}(\Gamma; \chi)$ for arbitrary $\Gamma \in \mathfrak{M}_2$ and $\chi \in \mathfrak{R}_s(\Gamma)$ (see [66]). As early as his lectures [50], Selberg indicated that, from a formula of which a more general version is now known as the Weyl-Selberg formula (see Theorem 4.4.1), one cannot, in general, extract any information concerning the asymptotic behavior of the distribution function for the eigenvalues of the discrete spectrum of $\mathfrak{A}(\Gamma; \chi)(\chi = 1)$, except in certain cases of arithmetic groups Γ for which one can explicitly compute the corresponding determinants of the scattering matrices in terms of the Riemann zeta-function and other special functions of analytic number theory. And in second part of §6.1 we consider the examples of arithmetic groups (congruence-subgroups) for which the Weyl-Selberg formula and explicit formulas for the determinants of the scattering matrices imply Roelcke's conjecture (and in a significantly stronger form). More precisely, in Theorem 6.1.2 we establish Weyl's formula for the eigenvalues of the discrete spectrum of $\mathfrak{A}(\Gamma; 1)$ in the case when Γ is a congruence-subgroup $\Gamma_1(m)$ or $\Gamma_2(m)$; and this result is made even stronger in Theorem 6.1.1 for a congruence-subgroup $\Gamma_0(m)$.

In §6.2 we derive a formula for the Selberg zeta-function of a compact Riemann surface (Theorem 6.2.3); this formula should be regarded as a transcendental analog of the Artin-Takagi formula in algebraic number theory. The ground field in this situation corresponds to an arbitrary normal subgroup of finite index in the fundamental group of an arbitrary compact Riemann surface of genus no less than two. Our formula is obtained as a consequence of a more general theory for the resolvent of the operator $\mathfrak{A}(\Gamma; \chi)$, a theory which holds for any group $\Gamma \in \mathfrak{M}$ and any representation $\chi \in \mathfrak{R}(\Gamma)$ (Theorems 6.2.1 and 6.2.2). Another consequence of this theory is the Roelcke conjecture. We prove that for every group $\Gamma \in \mathfrak{M}_2$ there exists a subgroup of finite index $\Gamma_1 \subset \Gamma$ such that the distribution function of the eigenvalues of the discrete spectrum of $\mathfrak{A}(\Gamma_1; 1)$ is unbounded at infinity. We also give a lower bound with an effective constant for this distribution function in any sufficiently long finite interval. All of these results were first published in [67].

The basic purpose of §6.3 is to deepen the spectral theory of $\mathfrak{A}(\Gamma; \chi)$ in the case of special Fuchsian groups of the first kind Γ—groups with nontrivial commensurables. The basic results are 1) construction of a simultaneous spectral decomposition for the operator $\mathfrak{A}(\Gamma;\chi)$ and the Hecke operator $T(g)$ (Theorems 6.3.3–6.3.5), and 2) proof of Roelcke's conjecture for a group $\Gamma \in \mathfrak{M}_2$ with large commensurable (Theorem 6.3.6). These theorems were first published in [64]. Here we shall not consider the theory in its most general form, but shall limit ourselves to the trivial representation χ, $\dim V = 1$. At the end of the section we give many examples of groups with nontrivial commensurables. Among them the set of groups Γ_M occupies an especially important place. Each group $\Gamma_M \in \mathfrak{M}$ is a subgroup of index two in a group generated by reflections relative to the sides of a regular polygon M in the Lobachevsky plane (see §6.3). We note that the set of all groups commensurable with the groups Γ_M is rather extensive. In particular, this set contains all arithmetic subgroups $\Gamma \in \mathfrak{M}_2$ as a small subset.

In §6.4 the theory developed in §6.3 is specialized to the case of an arbitrary group Γ which is commensurable with one of the groups Γ_M. The basic results are a proof of the Roelcke conjecture for Γ ($\chi = 1$) (Theorem 6.4.5), a proof of a still stronger conjecture concerning the distribution function for the eigenvalues of the discrete spectrum of $\mathfrak{A}(\Gamma; 1)$ (Theorem 6.4.7), and, finally, the demonstration of a connection between the spectral theory of automorphic functions for an arbitrary group Γ_M and the Dirichlet and Neumann boundary value problems on M (Theorems 6.4.2–6.4.4 and 6.4.6). All of these theorems were first published in [61] and [70]. The proof of the special case of Roelcke's conjecture for Hecke groups and for $\chi = 1$ was given earlier by Roelcke himself in his dissertation [45].

§6.5 is devoted to a derivation of the trace formula for Dirichlet's problem from Theorem 6.4.6, which can be naturally regarded as a variant of the classical Selberg trace formula. At the beginning of the section we prove a spectral trace formula for Dirichlet's problem on an arbitrary regular polygon M. We then consider the theory separately for compact M (§6.5a)) and noncompact M (§6.5b)). The proof of the Selberg trace formula for Dirichlet's problem is based on an investigation of the relative conjugacy classes $\{\mathfrak{S}\gamma\}_{\Gamma_M}$, where $\mathfrak{S}$ is a fixed reflection relative to a side of M. In §6.5a) we pay particular attention to the nondegenerate classes $\{\mathfrak{S}_\gamma\}_{\Gamma_M}$ ($\operatorname{tr}\mathfrak{S}_\gamma \neq 0$), and in §6.5b) we look at the degenerate classes ($\operatorname{tr}\mathfrak{S}\gamma = 0$). As a simple consequence of our Selberg trace formula for Dirichlet's problem, in both the compact and noncompact cases we consider Weyl's asymptotic formula for the eigenvalues for the operator $\frac{1}{2}(\mathfrak{I} - T(\mathfrak{S}))\mathfrak{A}(\Gamma_M; 1)$ in Dirichlet's problem on M (the spectrum of this operator is purely discrete); this asymptotic formula is also proved in §6.5. We conclude the section by showing that the Selberg trace formula for the von Neumann problem on M is a consequence of the classical trace formula and the Selberg trace formula for Dirichlet's problem. The basic results of §6.5 were first published in the note [65] (see also [60], [62] and [66]).

In §6.6 we define the zeta-function $Z_M(s)$, which we call the *Selberg zeta-function* for the Dirichlet boundary value problem on a regular polygon M, and we prove its basic properties. Among them are meromorphicity and a functional equation. We give a complete description of all of the zeros and poles of the function. These results are all obtained from a fundamental representation for the logarithmic derivative of $Z_M(s)$ (Theorem 6.6.1), which, in turn, is a consequence of the Selberg trace formula for Dirichlet's problem in §6.5. The basic theorems of the section were first published in [65].