Basic Algebraic Geometry

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Preface

Algebraic geometry occupied a central place in the mathematics of the last century. The deepest results of Abel, Riemann, Weierstrass, many of the most important papers of Klein and Poincaré belong to this domain.

At the end of the last and the beginning of the present century the attitude towards algebraic geometry changed abruptly. Around 1910 Klein wrote:

"When I was a student, Abelian functions*—as an after-effect of Jacobi's tradition—were regarded as the undisputed summit of mathematics, and each of us, as a matter of course, had the ambition to forge ahead in this field. And now? The young generation hardly know what Abelian functions are." (Vorlesungen über die Entwicklung der Mathematik im XIX. Jahrhundert, Springer-Verlag, Berlin 1926, Seite 312).

The style of thinking that was fully developed in algebraic geometry at that time was too far removed from the set-theoretical and axiomatic spirit, which then determined the development of mathematics. Several decades had to lapse before the rise of the theory of topological, differentiable and complex manifolds, the general theory of fields, the theory of ideals in sufficiently general rings, and only then it became possible to construct algebraic geometry on the basis of the principles of set-theoretical mathematics.

Around the middle of the present century algebraic geometry had undergone to a large extent such a reshaping process. As a result, it can again lay claim to the position it once occupied in mathematics. The range of applicability of its ideas enlarged extraordinarily towards algebraic varieties over arbitrary fields and complex manifolds of the most general kind. Algebraic geometry, quite apart from many better achievements, succeeded in removing the charge of being "incomprehensible" and "unconvincing".

The basis for this rebuilding of algebraic geometry was algebra. In its first versions the use of a delicate algebraic apparatus often led

* From the present-day point of view, the theory of Abelian functions is the analytical aspect of the theory of projective algebraic group varieties.

to the disappearance of that vivid geometric style that was characteristic for the preceding period. However, the last two decades have brought many simplifications in the foundations of algebraic geometry, which have made it possible to come remarkably close to the ideal combination of logical transparency and geometrical intuitiveness.

The aim of the book is to set forth the elements of algebraic geometry to a fairly wide extent, so as to give a general idea of this branch of mathematics and to provide a basis for the study of the more specialist literature. The reader is not assumed to have any prior knowledge of algebraic geometry, neither of its general theorems nor of concrete examples. For this reason, side by side with the development of the general theory, applications and special cases take a prominent place, because they motivate new concepts and problems to be raised.

It seems to me that the logic of the subject will be clearer to the reader if in the spirit of the "biogenetic law" he repeats, in a very condensed way, the evolution of algebraic geometry. Therefore the very first section, for example, is devoted to the simplest properties of plane algebraic curves. Similarly, Part One of the book discusses only algebraic varieties situated in a projective space, and it is only in Part Two that the reader comes across schemes and the general concept of a variety.

Part Three is concerned with algebraic varieties over the complex field and their connections with complex analytic manifolds. In this part the reader needs some acquaintance with the elements of topology and the theory of analytic functions.

My sincere thanks are due to all who have helped me with their advice during the work on this book. It is based on notes of some courses I have given at the University of Moscow. Many members of the audience and many readers of these notes have made very useful comments to me. I am particularly indebted to the editor, B. G. Moishezon. Numerous conversations with him were very useful for me. A number of proofs spread over the book are based on his advice.

Advice to the Reader

The first two parts of the book assume very little knowledge on the part of the reader. This amounts to the contents of a university course in algebra and analytic geometry and the rudiments of the theory of fields, which the reader could find, for example, in any of the following books: van der Waerden, Algebra, Vol. I, Ch. V and VIII, Zariski and Samuel, Commutative algebra, Vol. I, Ch. II, or Lang, Algebra, Ch. VII and X. Apart from this, frequent use is made of Hilbert's Basis Theorem and Hilbert's Nullstellensatz. Proofs are contained in the book by Zariski and Samuel, Vol. I, Ch. IV, § 1 and Vol. II, Ch. II, § 3.

In addition, in a few places we use certain isolated results of commutative algebra, and for their proofs the reader is referred to the book by Zariski and Samuel. In all cases the matter concerns only a few pages, which can be understood independently of the remaining parts of the book.

The third part assumes more knowledge. Essentially this concerns topology. Singular homology theory is taken as known, as are properties of differential forms, and Stokes' theorem. In Ch. VII, § 1 the concept of a differentiable manifold is applied, also Poincaré's duality law, and some properties of intersections of cycles; in §§ 3 and 4 of the same chapter we use the combinatorial classification of surfaces, but these three sections are not necessary for an understanding of the rest of the book. In the last section of the book we use one result from Morse theory, which can, however, simply be taken on trust. Finally, in the third part the reader is assumed to be familiar with the elements of the theory of analytic functions—a standard university course is amply sufficient.

The second and third parts of the book are based on the first. However, there are passages in it that are not needed for an understanding of what follows. They are Ch. IV, §§ 2 and 3, Ch. I, §§ 6.4 and 6.5, Ch. II, § 1.5, Ch. II, § 4.5, Ch. II, § 5.5, Ch. III, §§ 5.6 and 5.7; Ch. III, § 3 is fairly isolated: it is connected only with Ch. VIII, § 1.3.

The reader who is interested only in varieties over the field of complex numbers, and altogether in the more "classical" aspects of algebraic geometry, might study Ch. V only superficially. Finally, there are a number of places where we report without proofs on further developments of the questions considered in the book. Of course, these passages are not essential for an understanding of what follows.

In conclusion I wish to indicate the literature that has a bearing on the problems treated in the book and can form the nucleus of a further intensive study of algebraic geometry.

Every reader who is interested in algebraic geometry simply has to study the cohomology theory of algebraic coherent sheaves. Within the framework of the theory of varieties one can become acquainted with it through Serre's paper "Faisceaux algébriques cohérents", Ann. of Math. (2) **61**, 197–278 (1955), or Zariski's "Algebraic sheaf theory", Bull. Amer. Math. Soc. **62**, 117–141 (1956). Within the framework of the theory of schemes there is an account of the theory in the notes of Manin's "Lectures on algebraic geometry", Moscow State University 1968. A natural continuation of this theory is the general Riemann-Roch theorem, which can be read up in the paper by Borel and Serre, "Le théorème de Riemann-Roch", Bull. Soc. Math. France **86**, 97–136 (1958), or Manin's "Lectures on the K-functor in algebraic geometry" [Uspekhi Mat. Nauk **24**:5, 3–86 (1969) = Russian Math. Surveys **24**:5, 1–89 (1969)].

In this book there are frequent references to the Riemann-Roch theorem for curves, but it is never proved. Of course, it follows from the general Riemann-Roch theorem, but it can also be easily derived directly from general properties of the cohomology of algebraic coherent sheaves. Such a proof can be found in the book by Serre "Groupes algébriques et corps de classes", Hermann et Cie., Paris 1959, Ch. II. One can become acquainted with the theory of algebraic surfaces in the book "Algebraic surfaces", Trudy Mat. Inst. Steklov **75** (1965).

The elements of the theory of algebraic groups can be found in Borel, "Groupes linéaires algébriques", Ann. of Math. (2) **64**, 20–82 (1956), or Mumford, "Abelian varieties", Oxford University Press, London 1970.

So far there are no accounts of the general theory of schemes having the character of a textbook. Mumford's mimeographed lecture notes "Introduction to algebraic geometry", Harvard University Notes, can serve as an excellent introduction, and a very full account is in Grothendieck and Dieudonné's many-volume work "Eléments de géometrie algébrique". (Vol. I, Springer-Verlag, Berlin-Heidelberg-New York 1971), which is not yet completed.

In our book the number-theoretical aspect of algebraic geometry is almost nowhere touched upon, although this aspect played a very important role in the development of this branch of mathematics and several of the most brilliant applications are connected with it. An idea of this circle of problems can be obtained from Lang's "Diophantine geometry", Interscience, New York-London 1962, and Cassel's paper "Diophantine equations with special reference to elliptic curves", J. London Math. Soc. **41**, 193–291 (1966).

One can become acquainted with the "analytic" direction in the theory of algebraic varieties and the closely related theory of analytic manifolds in the book by Weil "Introduction à l'étude des variétés Kählériennes", Hermann et Cie., Paris 1958, and Chern's "Complex manifolds", Univ. of Recife, 1959.

Finally, a great help in the understanding of algebraic geometry is familiarity with the works of the classical, above all the Italian geometers. Of the vast literature I only mention a few works that are least specialized: F. Enriques and O. Chisini, "Lezioni sulla teoria geometrica delle equazione e delle funzione algebriche, 3 vols., Bologna 1915–1924; G. Castelnuovo and F. Enriques, "Die algebraischen Flächen vom Gesichtspunkte der birationalen Transformationen aus", Enzykl. d. math. Wiss., III, 3; F. Severi "Vorlesungen über algebraische Geometrie", Leipzig 1921; O. Zariski, "Algebraic surfaces", second ed. Springer-Verlag, Berlin-Heidelberg-New York 1971 (the basic text of the book contains an account of the classical papers, and the appendix their translation into the language of present-day concepts).

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Part I. Algebraic Varieties in a Projective Space

Chapter I. Fundamental Concepts

§ 1. Plane Algebraic Curves

The first chapter is concerned with a number of fundamental concepts of algebraic geometry. In the first section we analyse some examples, which prepare us for the introduction of these concepts.

1. Rational Curves. The curve given by the equation

$$y^2 = x^2 + x^3 \tag{1}$$

has the property that the coordinates of its points can be expressed as rational functions of a parameter. To derive this expression we observe that the line $y = tx$ through the origin of coordinates intersects the curve (1), apart from the origin, in a single point. For by substituting $y = tx$ in (1) we obtain $x^2(t^2 - x - 1) = 0$. The root $x = 0$ corresponds to the point $O = (0, 0)$. Apart from this we have one other root $x = t^2 - 1$. From the equation of the line we find that $y = t(t^2 - 1)$. Thus, we have the required parameterization

$$x = t^2 - 1, \quad y = t(t^2 - 1), \tag{2}$$

and we have also clarified its geometric meaning: t is the slope of the line passing through the points (x, y) and O, and x and y corresponding to t are the coordinates of the point of intersection, other than O, of the line $y = tx$ and the curve (1). We can represent this parameterization even more intuitively by drawing any line not passing through O (for example, the line with the equation $x = 1$) and associating with a point P the point of intersection Q of the line OP with the chosen line (projection of the curve from O) (Fig. 1). Here the coordinate on the chosen line plays the role of the parameter t. Both from this geometric interpretation and from (2) it is clear that the parameter t is uniquely determined (for $x \neq 0$) by the point (x, y).

Now we give a general definition of plane algebraic curves for which such a representation is possible. As a preliminary we introduce some concepts. We fix a field k. Henceforth we mean by points the points of the (x, y)-plane whose coordinates belong to k.