Essential Concepts of **Differential Equations**

Volume II

Calanthia Wright

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Edited by **Calanthia Wright**

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Preface

A differential equation is an integral part of the vast field of mathematics. It can be defined as a mathematical equation that relates some function of one or more variables with its derivatives. The mathematical theory of differential equations can be said to have developed together with the sciences where the equations had derived from and where the results found application or were needed. Differential equations arise whenever a deterministic relation concerning some constantly changing quantities and their rates of change in space and time is known or hypothesized. Such relations are extremely familiar and therefore differential equations play a fundamental role in many disciplines like physics, engineering, biology and economics. The mathematical theory behind the equations can also be viewed as a uniting principle behind various phenomena. The theory of conduction of heat is one of the examples of a phenomena governed by a differential equation, that is, the heat equation. One will find that there are many processes that, while seemingly different, are described by differential equations. Diverse problems, sometimes stemming from quite distinct scientific fields, may give rise to identical differential equations. Many fundamental laws of physics and chemistry can be formulated as differential equations. Even in fields such as biology and economics, differential equations can be used to represent the behavior of complex systems. Thus the arena of differential equations can be said to be quite a prolific one.

This book is an attempt to compile and collate all available research on the subject of differential equations under one umbrella. I am grateful to those who put their hard work, effort and expertise into these researches as well as those who were supportive in this endeavour. I also wish to thank my publisher for giving me this unmatched opportunity. I am extremely thankful to all the contributing authors who took out their precious time to interact with me and helped me understand their research perspectives in a better manner for the best output. Lastly, I wish to thank my family for their constant support.

Editor

Contents

An Extension of the Optimal Homotopy Asymptotic Method to Coupled Schrödinger-KdV Equation

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We consider the approximate solution of the coupled Schrödinger-KdV equation by using the extended optimal homotopy asymptotic method (OHAM). We obtained the extended OHAM solution of the problem and compared with the exact, variational iteration method (VIM) and homotopy perturbation method (HPM) solutions. The obtained solution shows that extended OHAM is effective, simpler, easier, and explicit and gives a suitable way to control the convergence of the approximate solution.

1. Introduction

The nonlinear Schrödinger equations are of great interest due to their numerous applications in physical phenomena. The coupled Schrödinger-KdV equations are extensively used to model nonlinear dynamics of one-dimensional Langmuir and ion acoustic waves in the system of coordinates moving at the speed of ion acoustic. This problem remains under consideration from many years and has been investigated by many researchers. Many authors have investigated the nonlinear Schrödinger-KdV equation by various techniques such as the following: Wang [1] used finite difference method, Küçükarslan [2] used HPM, Bai and Zhang [3] used quadratic B-Spline finite element method, Fan and Hon [4] used extended tanh method, Kaya and El-Sayed [5] used adomian decomposition method (ADM), Doosthoseini and Shahmohamadi [6] used VIM, Alomari et al. [7] used homotopy analysis method (HAM), Qing et al. [8] used element free Galerkin method (EFG), and Golbabai and Safdari-Vaighani [9] used meshless method using RBF collocation scheme. The perturbation methods like HPM required a small parameter and are difficult to determine.

Recently, Marinca et al. introduced OHAM [10–14] for the solution of nonlinear problems which made the perturbation methods independent of the assumption of small parameters and huge computational work.

The motivation of this paper is to extend the OHAM formulation for a system of three partial differential equations and to apply the extended OHAM formulation to coupled nonlinear Schrödinger-KdV equation. In [15–17] OHAM has been proved to be valuable for obtaining an approximate solution of ordinary/partial differential equations (O/PDEs). Before, this system of nonlinear partial differential equations (NPDEs) was not solved by OHAM. We have proved that extended OHAM is useful and reliable for NPDEs, showing its validity and great potential for the solution of transient physical phenomenon in science and engineering.

In the succeeding section, the basic idea of extended OHAM is formulated for the solution of system NPDEs. The effectiveness and efficiency of OHAM are shown in Section 3.

2. Extended Mathematical Formulation of OHAM

Consider a system of three partial differential equations:

$$\mathscr{A}_1\left(f\left(x,t\right)\right)+s_1\left(x,t\right)=0,$$

$$\mathscr{A}_2\left(g\left(x,t\right)\right)+s_2\left(x,t\right)=0,$$

$$\mathscr{A}_3\left(h\left(x,t\right)\right)+s_2\left(x,t\right)=0,$$

$$x\in\Omega$$

$$\mathscr{B}_1\left(f, \frac{\partial f}{\partial x}\right) = 0,$$
$$\mathscr{B}_2\left(g, \frac{\partial g}{\partial x}\right) = 0,$$
$$\mathscr{B}_3\left(h, \frac{\partial h}{\partial x}\right) = 0,$$
$$x \in \Gamma, \tag{1}$$

where $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ are differential operators, $f(x,t), g(x,t), h(x,t)$ are unknown functions, x and t denote spatial and temporal independent variables, respectively, Γ is the boundary of Ω, and $s_1(x,t), s_2(x,t), s_3(x,t)$ are known analytic functions. $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ can be divided into two parts:

$$\begin{aligned} \mathscr{A}_1 &= \mathscr{L}_1 + \mathscr{N}_1, \\ \mathscr{A}_2 &= \mathscr{L}_2 + \mathscr{N}_2, \\ \mathscr{A}_3 &= \mathscr{L}_3 + \mathscr{N}_3. \end{aligned} \tag{2}$$

$\mathscr{L}_1, \mathscr{L}_2, \mathscr{L}_3$ contain the linear parts while $\mathscr{N}_1, \mathscr{N}_2, \mathscr{N}_3$ contain the nonlinear parts of the system of partial differential equations.

According to OHAM, we construct

$$\begin{aligned} \alpha(x,t;p) &: \Omega \times [0,1] \longrightarrow \mathfrak{R}, \\ \beta(x,t;p) &: \psi \times [0,1] \longrightarrow \mathfrak{R}, \\ \gamma(x,t;p) &: \phi \times [0,1] \longrightarrow \mathfrak{R}, \end{aligned} \tag{3}$$

satisfying the following homotopies:

$$\begin{aligned} &H(\alpha(x,t;p),p) \\ &\quad = (1-p)\{\mathscr{L}_1(\alpha(x,t;p)) + s_1(x,t)\} \\ &\qquad - H_1(p)\{\mathscr{A}_1(\alpha(x,t;p)) + s_1(x,t)\} = 0, \\ &H(\beta(x,t;p),p) \\ &\quad = (1-p)\{\mathscr{L}_2(\beta(x,t;p)) + s_2(x,t)\} \\ &\qquad - H_2(p)\{\mathscr{A}_2(\beta(x,t;p)) + s_2(x,t)\} = 0, \\ &H(\gamma(x,t;p),p) \\ &\quad = (1-p)\{\mathscr{L}_3(\gamma(x,t;p)) + s_3(x,t)\} \\ &\qquad - H_3(p)\{\mathscr{A}_3(\gamma(x,t;p)) + s_3(x,t)\} = 0, \end{aligned} \tag{4}$$

where the auxiliary functions $H_1(p), H_2(p), H_3(p)$ are nonzero for $p \neq 0$ and $H_1(0) = 0, H_2(0) = 0, H_3(0) = 0$. Equation (4) is called optimal homotopy equation. Clearly, we have

$$\begin{aligned} &p = 0 \Longrightarrow H(\alpha(x,t;0),0) = \mathscr{L}_1(\alpha(x,t;0)) + s_1(x,t) = 0, \\ &p = 0 \Longrightarrow H(\beta(x,t;0),0) = \mathscr{L}_2(\beta(x,t;0)) + s_2(x,t) = 0, \\ &p = 0 \Longrightarrow H(\gamma(x,t;0),0) = \mathscr{L}_3(\gamma(x,t;0)) + s_3(x,t) = 0, \\ &p = 1 \Longrightarrow H(\alpha(x,t;1),1) \\ &\quad = H_1(1)\{\mathscr{A}_1(\alpha(x,t;p)) + s_1(x,t)\} = 0, \\ &p = 1 \Longrightarrow H(\beta(x,t;1),1) \\ &\quad = H_2(1)\{\mathscr{A}_2(\beta(x,t;p)) + s_2(x,t)\} = 0, \\ &p = 1 \Longrightarrow H(\gamma(x,t;1),1) \\ &\quad = H_3(1)\{\mathscr{A}_3(\gamma(x,t;p)) + s_3(x,t)\} = 0. \end{aligned} \tag{5}$$

Obviously, when $p = 0$ and $p = 1$ we obtain

$$\begin{aligned} &\alpha(x,t;0) = f_0(x,t), \beta(x,t;0) \\ &\quad = g_0(x,t), \gamma(x,t;0) = h_0(x,t), \\ &\alpha(x,t;1) = f(x,t), \beta(x,t;1) \\ &\quad = g(x,t), \gamma(x,t;1) = h(x,t), \end{aligned} \tag{6}$$

respectively. When p varies from 0 to 1, the solution $\alpha(x,t;p), \beta(x,t;p), \gamma(x,t;p)$ approaches from $f_0(x,t), g_0(x,t), h_0(x,t)$ to $f(x,t), g(x,t), h(x,t)$, where $f_0(x,t), g_0(x,t), h_0(x,t)$ are obtained from (4) for $p = 0$:

$$\begin{aligned} &\mathscr{L}_1(f_0(x,t)) + s_1(x,t) = 0, \qquad \mathscr{B}_1\left(f_0, \frac{\partial f_0}{\partial x}\right) = 0, \\ &\mathscr{L}_2(g_0(x,t)) + s_2(x,t) = 0, \qquad \mathscr{B}_2\left(g_0, \frac{\partial g_0}{\partial x}\right) = 0, \\ &\mathscr{L}_3(h_0(x,t)) + s_3(x,t) = 0, \qquad \mathscr{B}_3\left(h_0, \frac{\partial h_0}{\partial x}\right) = 0. \end{aligned} \tag{7}$$

We choose auxiliary functions $H_1(p), H_2(p), H_3(p)$ in the form

$$\begin{aligned} H_1(p) &= pC_{11} + p^2C_{12} + p^3C_{13} + \cdots + p^mC_{1m}, \\ H_2(p) &= pC_{21} + p^2C_{22} + p^3C_{23} + \cdots + p^mC_{2m}, \\ H_3(p) &= pC_{31} + p^2C_{32} + p^3C_{33} + \cdots + p^mC_{3m}. \end{aligned} \tag{8}$$

To get the approximate solutions, we expand $\alpha(x,t;p,C_{1i}), \beta(x,t;p,C_{2i}), \gamma(x,t;p,C_{3i})$ by Taylor's series about p in the following manner:

$$\begin{aligned} \alpha(x,t;p,C_{1i}) &= f_0(x,t) + \sum_{k\geq 1} f_k(x,t;C_{1i})\ p^k, \\ \beta(x,t;p,C_{2i}) &= g_0(x,t) + \sum_{l\geq 1} g_l(x,t;C_{2i})\ p^k, \\ \gamma(x,t;p,C_{3i}) &= h_0(x,t) + \sum_{n\geq 1} h_n(x,t;C_{3i})\ p^k, \end{aligned} \tag{9}$$

where $k = l = n = i = 1, 2, 3, 4, \ldots$. Now substituting (8)-(9) into (4) and equating the coefficient of like powers of p, we obtain zeroth order system, given by (7) and the first and second order systems given by (10)-(11), respectively, and the general governing equations for $u_k(x,t)$ are given by (12):

$$\begin{aligned}
&\mathscr{L}_1(f_1(x,t)) - \mathscr{L}_1(f_0(x,t)) \\
&\quad = C_{11}(\mathscr{L}_1(f_0(x,t)) + \mathscr{N}_1(f_0(x,t))), \\
&\mathscr{B}_1\left(f_1, \frac{\partial f_1}{\partial x}\right) = 0 \\
&\mathscr{L}_2(g_1(x,t)) - \mathscr{L}_2(g_0(x,t)) \\
&\quad = C_{21}(\mathscr{L}_2(g_0(x,t)) + \mathscr{N}_2(g_0(x,t))), \\
&\mathscr{B}_2\left(g_1, \frac{\partial g_1}{\partial x}\right) = 0 \\
&\mathscr{L}_3(h_1(x,t)) - \mathscr{L}_3(h_0(x,t)) \\
&\quad = C_{31}(\mathscr{L}_3(h_0(x,t)) + \mathscr{N}_3(h_0(x,t))), \\
&\mathscr{B}_3\left(h_1, \frac{\partial h_1}{\partial x}\right) = 0
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\mathscr{L}_1(f_2(x,t)) - \mathscr{L}_1(f_1(x,t)) \\
&\quad = C_{11}(\mathscr{L}_1(f_1(x,t)) + \mathscr{N}_1(f_0(x,t), f_1(x,t))) \\
&\qquad + C_{12}(\mathscr{L}_1(f_0(x,t)) + \mathscr{N}_1(f_0(x,t))), \\
&\mathscr{B}_1\left(f_2, \frac{\partial f_2}{\partial x}\right) = 0, \\
&\mathscr{L}_2(g_2(x,t)) - \mathscr{L}_2(g_1(x,t)) \\
&\quad = C_{21}(\mathscr{L}_2(g_1(x,t)) + \mathscr{N}_2(g_0(x,t), g_1(x,t))) \\
&\qquad + C_{22}(\mathscr{L}_2(g_0(x,t)) + \mathscr{N}_2(g_0(x,t))), \\
&\mathscr{B}_2\left(g_2, \frac{\partial g_2}{\partial x}\right) = 0, \\
&\mathscr{L}_3(h_2(x,t)) - \mathscr{L}_3(h_1(x,t)) \\
&\quad = C_{31}(\mathscr{L}_3(h_1(x,t)) + \mathscr{N}_3(h_0(x,t), h_1(x,t))) \\
&\qquad + C_{32}(\mathscr{L}_3(h_0(x,t)) + \mathscr{N}_3(h_0(x,t))), \\
&\mathscr{B}_3\left(h_2, \frac{\partial h_2}{\partial x}\right) = 0,
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\mathscr{L}_1(f_3(x,t)) - \mathscr{L}_1(f_2(x,t)) \\
&\quad = C_{11}(\mathscr{L}_1(f_2(x,t)) \\
&\qquad + \mathscr{N}_1(f_0(x,t), f_1(x,t), f_2(x,t))) \\
&\qquad + C_{12}(\mathscr{L}_1(f_1(x,t)) + \mathscr{N}_1(f_0(x,t), f_1(x,t))) \\
&\qquad + C_{13}(\mathscr{L}_1(f_0(x,t)) + \mathscr{N}_1(f_0(x,t))) \\
&\mathscr{B}_1\left(f_3, \frac{\partial f_3}{\partial x}\right) = 0, \\
&\mathscr{L}_2(g_3(x,t)) - \mathscr{L}_2(g_2(x,t)) \\
&\quad = C_{21}(\mathscr{L}_2(g_2(x,t)) \\
&\qquad + \mathscr{N}_2(g_0(x,t), g_1(x,t), g_2(x,t))) \\
&\qquad + C_{22}(\mathscr{L}_2(g_1(x,t)) + \mathscr{N}_2(g_0(x,t), g_1(x,t))) \\
&\qquad + C_{23}(\mathscr{L}_2(g_0(x,t)) + \mathscr{N}_2(g_0(x,t))), \\
&\mathscr{B}_2\left(g_3, \frac{\partial g_3}{\partial x}\right) = 0, \\
&\mathscr{L}_3(h_3(x,t)) - \mathscr{L}_3(h_2(x,t)) \\
&\quad = C_{31}(\mathscr{L}_3(h_2(x,t)) \\
&\qquad + \mathscr{N}_3(h_0(x,t), h_1(x,t), h_2(x,t))) \\
&\qquad + C_{32}(\mathscr{L}_3(h_1(x,t)) + \mathscr{N}_3(h_0(x,t), h_1(x,t))) \\
&\qquad + C_{33}(\mathscr{L}_3(h_0(x,t)) + \mathscr{N}_3(h_0(x,t))), \\
&\mathscr{B}_3\left(h_3, \frac{\partial h_3}{\partial x}\right) = 0,
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\mathscr{L}_1(f_k(x,t)) - \mathscr{L}_1(f_{k-1}(x,t)) \\
&\quad = \sum_{i=1}^{k} C_{1i}[\mathscr{L}_1(f_{k-i}(x,t)) \\
&\qquad + \mathscr{N}_1(f_0(x,t), f_1(x,t), \ldots, f_{k-i}(x,t))], \\
&\qquad k = 2, 3, \ldots, \\
&\mathscr{B}_1\left(f_k, \frac{\partial f_k}{\partial x}\right) = 0, \\
&\mathscr{L}_2(g_k(x,t)) - \mathscr{L}_2(g_{k-1}(x,t)) \\
&\quad = \sum_{i=1}^{k} C_{2i}[\mathscr{L}_2(g_{k-i}(x,t)) \\
&\qquad + \mathscr{N}_2(g_0(x,t), g_1(x,t), \ldots, g_{k-i}(x,t))], \\
&\qquad k = 2, 3, \ldots, \\
&\mathscr{B}_2\left(g_k, \frac{\partial g_k}{\partial x}\right) = 0, \\
&\mathscr{L}_3(h_k(x,t)) - \mathscr{L}_3(h_{k-1}(x,t)) \\
&\quad = \sum_{i=1}^{k} C_{3i}[\mathscr{L}_3(h_{k-i}(x,t)) \\
&\qquad + \mathscr{N}_3(h_0(x,t), h_1(x,t), \ldots, h_{k-i}(x,t))], \\
&\qquad k = 2, 3, \ldots, \\
&\mathscr{B}_3\left(h_k, \frac{\partial h_k}{\partial x}\right) = 0.
\end{aligned} \tag{13}$$

It has been observed that the convergence of the series (9) depends upon the auxiliary constants $C_{11}, C_{12}, C_{13}, \ldots C_{21}, C_{22}, C_{23}, \ldots C_{31}, C_{32}, C_{33}, \ldots$. If it is convergent at $p = 1$, one has

$$\begin{aligned} \alpha^* (x,t;C_{1i}) &= f_0 (x,t) + \sum_{k\geq 1} f_k (x,t;C_{1i}), \\ \beta^* (x,t;C_{2i}) &= g_0 (x,t) + \sum_{l\geq 1} g_l (x,t;C_{2i}), \\ & \qquad i = 1,2,\ldots m \\ \gamma^* (x,t;C_{3i}) &= h_0 (x,t) + \sum_{n\geq 1} h_n (x,t;C_{3i}). \end{aligned} \tag{14}$$

Substituting (14) into (1.1), the following expression for residuals results:

$$\begin{aligned} R_1 (x,t;C_{1i}) &= \mathscr{L}_1 (\alpha^* (x,t;C_{1i})) \\ &\quad + s_1 (x,t) + \mathscr{N}_1 (\alpha^* (x,t;C_{1i})), \\ R_2 (x,t;C_{2i}) &= \mathscr{L}_2 (\beta^* (x,t;C_{2i})) \\ &\quad + s_2 (x,t) + \mathscr{N}_2 (\beta^* (x,t;C_{2i})), \\ R_3 (x,t;C_{3i}) &= \mathscr{L}_3 (\gamma^* (x,t;C_{3i})) \\ &\quad + s_3 (x,t) + \mathscr{N}_3 (\gamma^* (x,t;C_{3i})). \end{aligned} \tag{15}$$

If $R_1(x,t;C_{1i}) = 0$, $R_2(x,t;C_{2i}) = 0$, $R_3(x,t;C_{3i}) = 0$ then $\alpha^*(x,t;C_{1i})$, $\beta^*(x,t;C_{2i})$, and $\gamma^*(x,t;C_{3i})$ will be the exact solutions of the problem. Generally it does not happen, especially in nonlinear problems.

For the computation of auxiliary constants, C_{1i}, C_{2i}, C_{3i}, $i = 1,2,\ldots,m$, there are different methods like Galerkin's method, Ritz method, least squares method, and collocation method. One can apply the method of least squares as follows:

$$\begin{aligned} J_1 (C_{1i}) &= \int_0^t \int_\Omega R_1^2 (x,t;C_{1i})\, dx\, dt, \\ J_2 (C_{2i}) &= \int_0^t \int_\psi R_2^2 (x,t;C_{2i})\, dx\, dt, \\ J_3 (C_{3i}) &= \int_0^t \int_\phi R_3^2 (x,t;C_{3i})\, dx\, dt, \end{aligned} \tag{16}$$

$$\begin{aligned} \frac{\partial J_1}{\partial C_{11}} &= \frac{\partial J_1}{\partial C_{12}} = \cdots \frac{\partial J_1}{\partial C_{1m}} = \frac{\partial J_2}{\partial C_{21}} = \frac{\partial J_2}{\partial C_{22}} = \cdots \frac{\partial J_2}{\partial C_{2m}} \\ &= \frac{\partial J_3}{\partial C_{31}} = \frac{\partial J_3}{\partial C_{32}} = \cdots = \frac{\partial J_3}{\partial C_{3m}} = 0. \end{aligned} \tag{17}$$

The mth order approximate solution can be obtained by these constants so-obtained. The more general auxiliary functions $H_1(p), H_2(p), H_3(p)$ are useful for convergence, which depends upon constants $C_{11}, C_{12}, C_{21}, C_{22}, C_{31}, C_{32}, \ldots$, can be optimally identified by (17), and is useful in error minimization.

3. Application of Extended OHAM to Coupled Schrödinger-KdV Equation

To demonstrate the effectiveness of the extended OHAM for coupled Schrödinger- KdV equation taken from [6], we have

$$\frac{\partial \beta (x,t)}{\partial t} - \frac{\partial^2 \eta (x,t)}{\partial x^2} - \eta (x,t) \mu (x,t) = 0, \tag{18}$$

$$\begin{aligned} &\frac{\partial \eta (x,t)}{\partial t} + \frac{\partial^2 \beta (x,t)}{\partial x^2} + \beta (x,t) \mu (x,t) = 0, \\ &\frac{\partial \mu (x,t)}{\partial t} + 6\mu (x,t) \frac{\partial \mu (x,t)}{\partial x} + \frac{\partial^3 \mu (x,t)}{\partial x^3} \\ &\quad - 2\beta (x,t) \frac{\partial \mu (x,t)}{\partial x} - 2\eta (x,t) \frac{\partial \eta (x,t)}{\partial x} = 0, \end{aligned} \tag{19}$$

with boundary conditions

$$\begin{aligned} \beta (x,0) &= \cos (x), \\ \eta (x,0) &= \sin (x), \\ \mu (x,0) &= \frac{3}{4}. \end{aligned} \tag{20}$$

The exact solution of (19) for $-3 \leq x \leq 3$ and $0 \leq t \leq 1$ is given by

$$\begin{aligned} \beta (x,t) &= \cos \left(x + \frac{t}{4} \right), \\ \eta (x,t) &= \sin \left(x + \frac{t}{4} \right), \\ \mu (x,t) &= \frac{3}{4}. \end{aligned} \tag{21}$$

Applying the extended OHAM technique discussed in Section 2,

$$\begin{aligned} &(1-p) \frac{\partial \beta (x,t)}{\partial t} \\ &\quad - H_1 (p) \left[\frac{\partial \beta (x,t)}{\partial t} - \frac{\partial^2 \eta (x,t)}{\partial x^2} - \eta (x,t) \mu (x,t) \right] = 0, \\ &(1-p) \frac{\partial \eta (x,t)}{\partial t} \\ &\quad - H_2 (p) \left[\frac{\partial \eta (x,t)}{\partial t} + \frac{\partial^2 \beta (x,t)}{\partial x^2} + \beta (x,t) \mu (x,t) \right] = 0, \\ &(1-p) \frac{\partial \mu (x,t)}{\partial t} \\ &\quad - H_3 (p) \left[\frac{\partial \mu (x,t)}{\partial t} + 6\mu (x,t) \frac{\partial \mu (x,t)}{\partial x} + \frac{\partial^3 \mu (x,t)}{\partial x^3} \right. \\ &\qquad \left. -2\beta (x,t) \frac{\partial \mu (x,t)}{\partial x} - 2\eta (x,t) \frac{\partial \eta (x,t)}{\partial x} \right] \\ &= 0. \end{aligned} \tag{22}$$

Table 1: Absolute error of OHAM solution $\beta(x,t)$ corresponding to the exact solution.

x	$t = 0.5$	$t = 0.2$	$t = 0.1$	$t = 0.01$	$t = 0.001$
−3	4.37771×10^{-3}	8.5962×10^{-4}	3.51139×10^{-4}	6.29194×10^{-5}	6.57034×10^{-6}
−2	1.83163×10^{-2}	8.0058×10^{-3}	4.12582×10^{-3}	4.24051×10^{-4}	4.25219×10^{-5}
−1	2.41704×10^{-2}	8.5651×10^{-3}	4.10724×10^{-3}	3.95312×10^{-4}	3.93790×10^{-5}
0	7.80233×10^{-3}	1.2497×10^{-3}	3.12484×10^{-4}	3.12500×10^{-6}	3.12500×10^{-8}
1	1.57391×10^{-2}	7.2147×10^{-3}	3.76957×10^{-3}	3.91935×10^{-4}	3.93452×10^{-5}
2	2.48101×10^{-2}	9.0460×10^{-3}	4.38590×10^{-3}	4.26652×10^{-4}	4.25479×10^{-5}
3	1.10708×10^{-3}	2.5600×10^{-3}	9.69852×10^{-4}	6.91069×10^{-5}	6.63222×10^{-6}

Table 2: Absolute error of OHAM solution $\eta(x,t)$ corresponding to the exact solution.

x	$t = 0.5$	$t = 0.2$	$t = 0.1$	$t = 0.01$	$t = 0.001$
−3	1.33126×10^{-4}	1.6026×10^{-4}	2.83128×10^{-5}	1.39272×10^{-6}	1.79217×10^{-7}
−2	7.19141×10^{-3}	1.12962×10^{-3}	2.77505×10^{-4}	2.07074×10^{-6}	4.87727×10^{-8}
−1	6.4398×10^{-3}	1.06041×10^{-3}	2.71500×10^{-4}	3.63037×10^{-6}	1.26513×10^{-7}
0	2.32524×10^{-4}	1.62664×10^{-5}	1.59445×10^{-5}	1.85225×10^{-6}	1.85483×10^{-7}
1	6.69107×10^{-3}	1.04283×10^{-3}	2.54331×10^{-4}	1.62882×10^{-6}	7.39210×10^{-8}
2	6.99788×10^{-3}	1.14315×10^{-3}	2.90776×10^{-4}	3.61236×10^{-6}	1.05604×10^{-7}
3	8.70868×10^{-4}	1.92467×10^{-4}	5.98826×10^{-5}	2.27472×10^{-6}	1.88037×10^{-7}

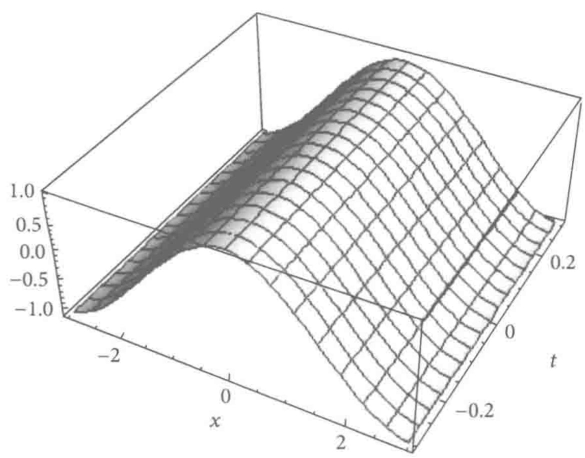

Figure 1: 3D, OHAM solution of $\beta(x,t)$ at $t = 0.1$.

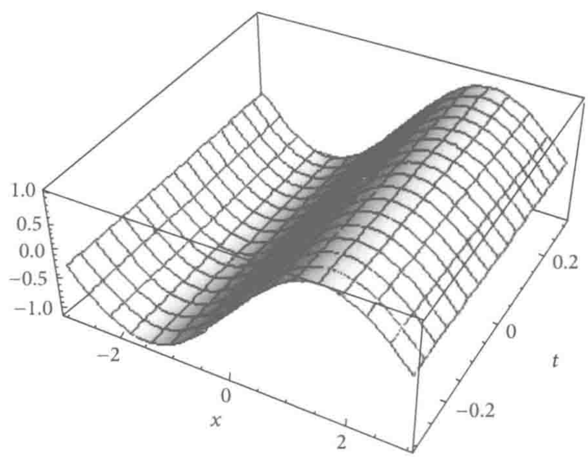

Figure 3: 3D, OHAM solution of $\eta(x,t)$ at $t = 0.1$.

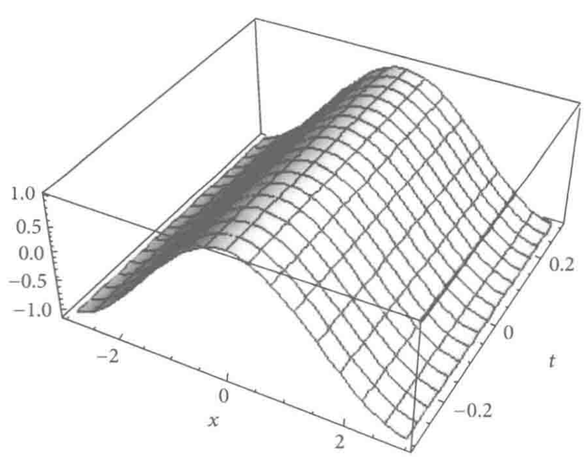

Figure 2: 3D, exact solution of $\beta(x,t)$ at $t = 0.1$.

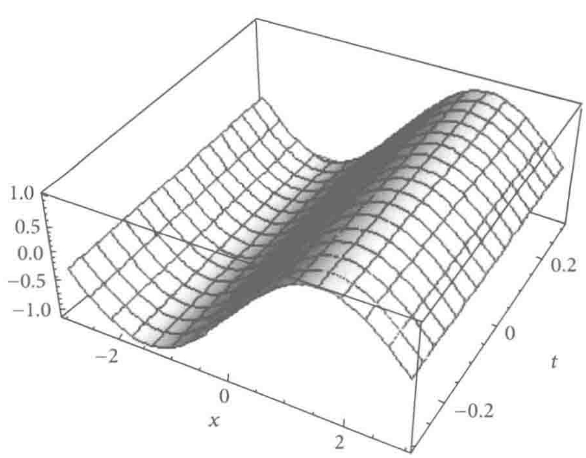

Figure 4: 3D, exact solution of $\eta(x,t)$ at $t = 0.1$.

TABLE 3: Comparison of $\mu(x,t)$ solutions obtained by OHAM to the exact solution.

x	OHAM solution	Exact solution
−3	3/4	3/4
−2	3/4	3/4
−1	3/4	3/4
0	3/4	3/4
1	3/4	3/4
2	3/4	3/4
3	3/4	3/4

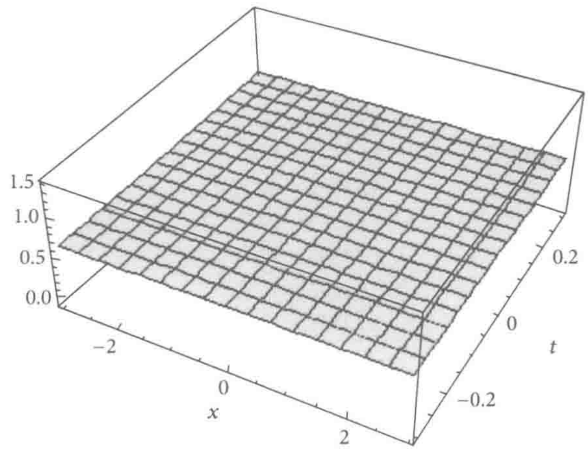

FIGURE 5: 3D, OHAM solution of $\mu(x,t)$ at $t=0.1$.

We consider

$$\beta=\beta_0+p\beta_1+p^2\beta_2,\qquad \eta=\eta_0+p\eta_1+p^2\eta_2,$$
$$\mu=\mu_0+p\mu_1+p^2\mu_2,$$
$$H_1(p)=pC_{11}+p^2C_{12},\tag{23}$$
$$H_2(p)=pC_{21}+p^2C_{22},\qquad H_3(p)=pC_{31}+p^2C_{32}.$$

Zeroth Order System. Consider

$$\frac{\partial\beta_0}{\partial t}=0,\qquad \frac{\partial\eta_0}{\partial t}=0,\qquad \frac{\partial\mu_0}{\partial t}=0,\tag{24}$$

with initial conditions

$$\beta_0(x,0)=\cos(x),$$
$$\eta_0(x,0)=\sin(x),\tag{25}$$
$$\mu_0(x,0)=\frac{3}{4}.$$

Its solution is

$$\beta_0(x,t)=\cos(x),$$
$$\eta_0(x,t)=\sin(x),\tag{26}$$
$$\mu_0(x,t)=\frac{3}{4}.$$

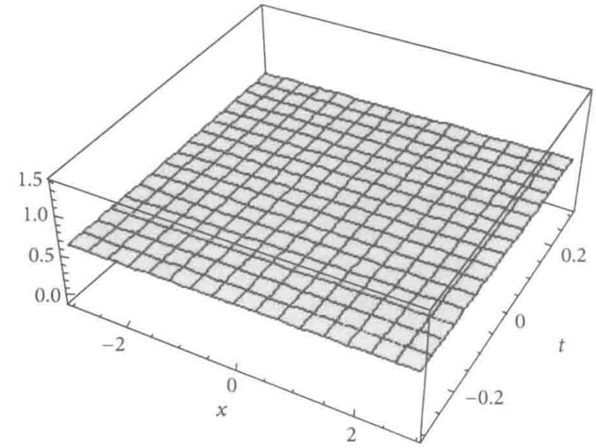

FIGURE 6: 3D, exact solution of $\mu(x,t)$ at $t=0.1$.

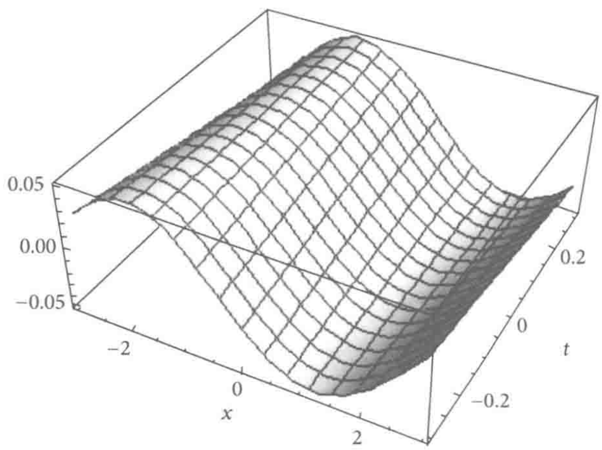

FIGURE 7: 3D, residual of $\beta(x,t)$ at $t=0.1$.

First Order System. Consider

$$\frac{\partial\beta_1(x,t)}{\partial t}=(1+C_{11})\frac{\partial\beta_0}{\partial t}-C_{11}\eta_0\mu_0-C_{11}\frac{\partial^2\eta_0}{\partial x^2},$$
$$\frac{\partial\eta_1(x,t)}{\partial t}=(1+C_{21})\frac{\partial\eta_0}{\partial t}+C_{21}\beta_0\mu_0+C_{21}\frac{\partial^2\beta_0}{\partial x^2},$$
$$\frac{\partial\mu_1(x,t)}{\partial t}=(1+C_{31})\frac{\partial\mu_0}{\partial t}\tag{27}$$
$$-2C_{31}\left(\beta_0\frac{\partial\beta_0}{\partial x}+\eta_0\frac{\partial\eta_0}{\partial x}-3\mu_0\frac{\partial\mu_0}{\partial x}\right)$$
$$+C_{31}\frac{\partial^3\mu_0}{\partial x^3}$$

with

$$\beta_1(x,0)=0,\qquad \eta_1(x,0)=0,\qquad \mu_1(x,0)=0.\tag{28}$$

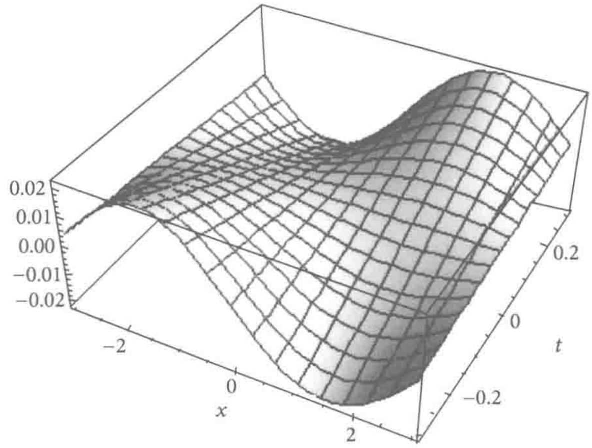

FIGURE 8: 3D, residual of $\eta(x,t)$ at $t = 0.1$.

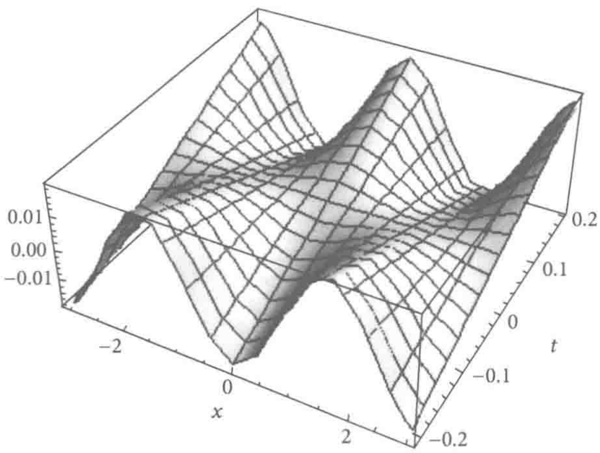

FIGURE 9: 3D, residual of $\mu(x,t)$ at $t = 0.1$.

Its solution is

$$\begin{aligned} \beta_1(x,t,C_{11}) &= \frac{C_{11}}{4} t \sin x, \\ \eta_1(x,t,C_{21}) &= -\frac{C_{21}}{4} t \cos x, \\ \mu_1(x,t,C_{31}) &= 0. \end{aligned} \tag{29}$$

Second Order System. Consider

$$\begin{aligned} \frac{\partial \beta_2(x,t)}{\partial t} &= \left[(1+C_{11}) \frac{\partial \beta_1}{\partial t} - C_{11} (\eta_1 \mu_0 + \eta_0 \mu_1) \right. \\ &\quad \left. + C_{12} (\beta_0 - \eta_0 \mu_0) - C_{12} \frac{\partial^2 \eta_0}{\partial x^2} - C_{11} \frac{\partial^2 \eta_1}{\partial x^2} \right], \\ \frac{\partial \eta_2(x,t)}{\partial t} &= \left[(1+C_{21}) \frac{\partial \eta_0}{\partial t} + C_{21} (\beta_1 \mu_0 + \beta_0 \mu_1) \right. \\ &\quad \left. + C_{22} (\eta_0 + \beta_0 \mu_0) + C_{21} \frac{\partial^2 \beta_1}{\partial x^2} + C_{22} \frac{\partial^2 \mu_0}{\partial x^2} \right], \end{aligned}$$

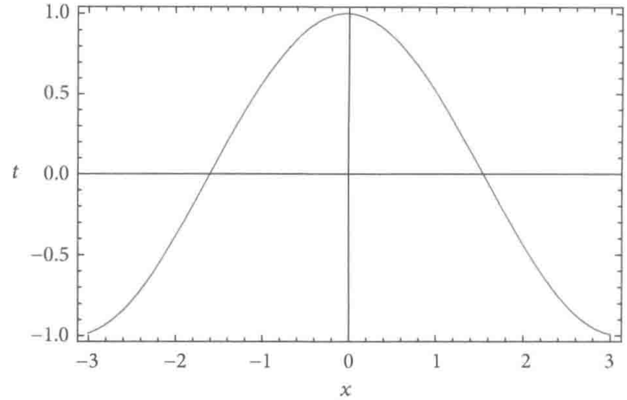

FIGURE 10: 2D, OHAM solution of $\beta(x,t)$ at $t = 0.1$.

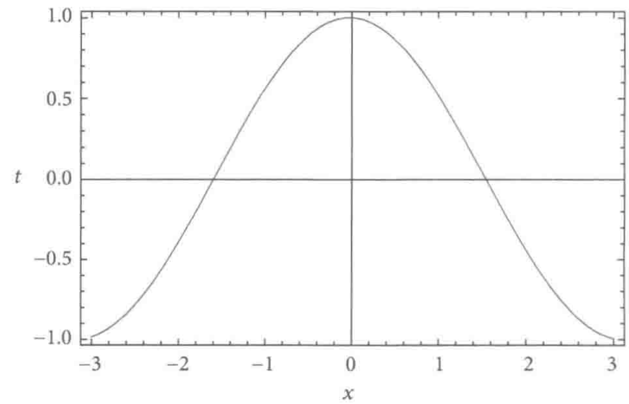

FIGURE 11: 2D, exact solution of $\beta(x,t)$ at $t = 0.1$.

$$\begin{aligned} \frac{\partial \mu_2(x,t)}{\partial t} &= \left[(1+C_{31}) \frac{\partial \mu_1}{\partial t} - 2C_{31} \right. \\ &\quad \times \left(\beta_1 \frac{\partial \beta_0}{\partial x} + \beta_0 \frac{\partial \beta_1}{\partial x} + \eta_1 \frac{\partial \eta_0}{\partial x} + \eta_0 \frac{\partial \eta_1}{\partial x} \right. \\ &\quad \left. - 3\mu_1 \frac{\partial \mu_0}{\partial x} - 3\mu_0 \frac{\partial \mu_1}{\partial x} \right) \\ &\quad - 2C_{32} \left(\beta_0 \frac{\partial \beta_0}{\partial x} + \eta_0 \frac{\partial \eta_0}{\partial x} - 3\mu_0 \frac{\partial \mu_0}{\partial x} \right) \\ &\quad \left. + C_{31} \frac{\partial \mu_0}{\partial x} + C_{32} \frac{\partial^3 \mu_0}{\partial x^3} + C_{31} \frac{\partial^3 \mu_1}{\partial x^3} \right] \end{aligned} \tag{30}$$

with

$$\beta_2(x,0) = 0, \qquad \eta_2(x,0) = 0, \qquad \mu_2(x,0) = 0. \tag{31}$$

The solution of second order system is

$$\begin{aligned} \beta_2(x,t,C_{11},C_{12}) &= \frac{1}{32} \left(8C_{11} t \sin x + 8C_{11}^2 t \sin x \right. \\ &\quad \left. + 8C_{12} t \sin x - t^2 C_{11} C_{21} \cos x \right), \end{aligned}$$

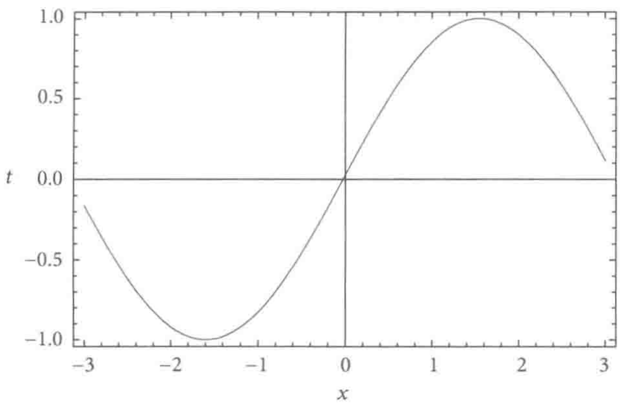

Figure 12: 2D, OHAM solution of $\eta(x,t)$ at $t = 0.1$.

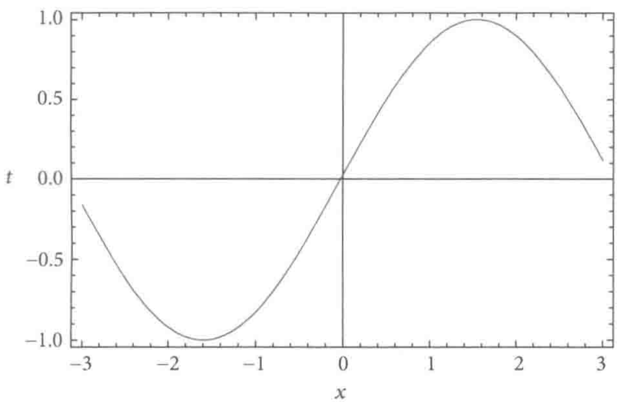

Figure 13: 2D, exact solution of $\eta(x,t)$ at $t = 0.1$.

$$\eta_2(x,t,C_{21},C_{22}) = -\frac{1}{32}\left(8C_{21}\cos x + 8C_{21}^2 t\cos x + C_{11}C_{21}t^2\sin x + C_{22}t\cos x\right),$$
$$\mu_2(x,t,C_{31}) = \frac{1}{4}\left(C_{21}C_{31}t^2\cos(2x) - C_{11}C_{31}t^2\cos(2x)\right). \quad (32)$$

Adding (26), (29), and (32), we obtain

$$\beta(x,t,C_{11},C_{21}) = \cos(x) + \frac{1}{4}C_{11}t\sin(x) + \frac{1}{32}\left(8C_{11}t\sin x + 8C_{11}^2 t\sin x + 8C_{12}t\sin x - t^2C_{11}C_{21}\cos x\right),$$

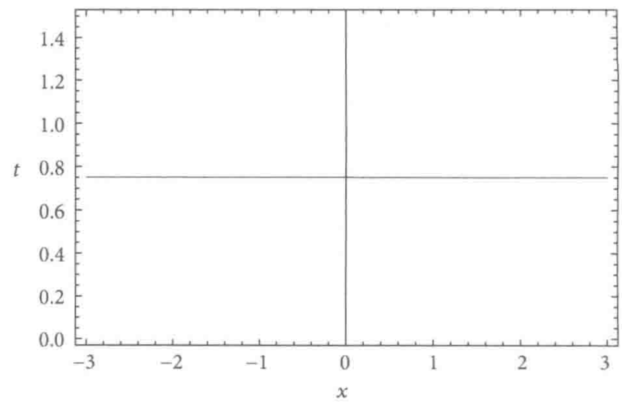

Figure 14: 2D, OHAM solution of $\mu(x,t)$ at $t = 0.1$.

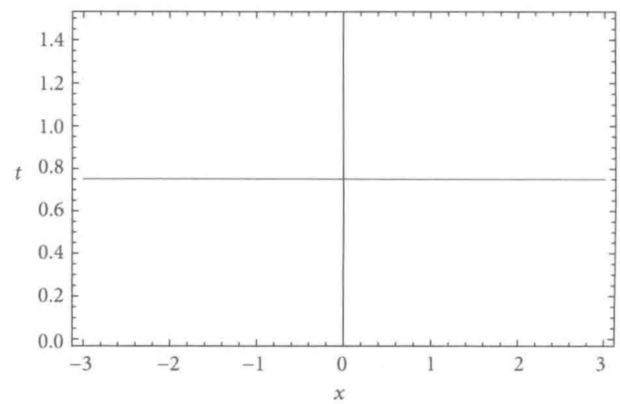

Figure 15: 2D, exact solution of $\mu(x,t)$ at $t = 0.1$.

$$\eta(x,t,C_{21},C_{22}) = \sin(x) - \frac{1}{4}C_{21}t\sin(x) - \frac{1}{32}\left(8C_{21}\cos x + 8C_{21}^2 t\cos x + C_{11}C_{21}t^2\sin x + C_{22}t\cos x\right),$$
$$\mu(x,t,C_{31}) = \frac{3}{4} + \frac{1}{4}\left(C_{21}C_{31}t^2\cos(2x) - C_{11}C_{31}t^2\cos(2x)\right). \quad (33)$$

For the calculation of the constants $C_{11}, C_{12}, C_{21}, C_{22}$, and C_{31} using (33) in (19) and applying the method of least square mentioned in (16)-(17) by taking, we get

$$\begin{aligned} C_{11} &= -3.041182429907255\times 10^{-14}, \\ C_{12} &= -1.1871110474593864 \\ C_{21} &= -3.041182429907255\times 10^{-14}, \\ C_{22} &= -0.999258572471839, \\ C_{31} &= -8.101774168020832\times 10^{-15}. \end{aligned} \quad (34)$$