Universitext

Jürgen Jost

Riemannian Geometry and Geometric Analysis

Sixth Edition

黎曼几何和几何分析 第6版

Springer

世界图书出版公司

www.wpcbj.com.cn

Jürgen Jost

Riemannian Geometry and Geometric Analysis

Springer

Jürgen Jost
Max Planck Institute
for Mathematics in the Sciences
Inselstr. 22
04103 Leipzig
Germany
jost@mis.mpg.de

ISBN 978-3-642-21297-0 e-ISBN 978-3-642-21298-7
DOI 10.1007/978-3-642-21298-7
Springer Heidelberg Dordrecht London New York

Library of Congress Control Number: 2011932682

Mathematics Subject Classification (2010): 53B21, 53L20, 32C17, 35I60, 49-XX, 58E20, 57R15

Reprint from English language edition:
Riemannian Geometry and Geometric Analysis Sixth Edition

by Jürgen Jost

Springer-Verlag Berlin Heidelberg is a part of Springer Science+Business Media

This reprint has been authorized by Springer Science & Business Media for distribution in China Mainland only and not for export therefrom.

Universitext

Universitext

Universitext is a series of textbooks that presents material from a wide variety of mathematical disciplines at master's level and beyond. The books, often well class-tested by their author, may have an informal, personal even experimental approach to their subject matter. Some of the most successful and established books in the series have evolved through several editions, always following the evolution of teaching curricula, to very polished texts.

Thus as research topics trickle down into graduate-level teaching, first textbooks written for new, cutting-edge courses may make their way into *Universitext*.

For further volumes:
www.springer.com/series/223

Dedicated to Shing-Tung Yau,
for so many discussions about
mathematics and Chinese culture

Preface

Riemannian geometry is characterized, and research is oriented towards and shaped by concepts (geodesics, connections, curvature, ...) and objectives, in particular to understand certain classes of (compact) Riemannian manifolds defined by curvature conditions (constant or positive or negative curvature, ...). By way of contrast, geometric analysis is a perhaps somewhat less systematic collection of techniques, for solving extremal problems naturally arising in geometry and for investigating and characterizing their solutions. It turns out that the two fields complement each other very well; geometric analysis offers tools for solving difficult problems in geometry, and Riemannian geometry stimulates progress in geometric analysis by setting ambitious goals.

It is the aim of this book to be a systematic and comprehensive introduction to Riemannian geometry and a representative introduction to the methods of geometric analysis. It attempts a synthesis of geometric and analytic methods in the study of Riemannian manifolds.

The present work is the sixth edition of my textbook on Riemannian geometry and geometric analysis. It has developed on the basis of several graduate courses I taught at the Ruhr-University Bochum and the University of Leipzig. The main new feature of the present edition is a systematic presentation of the spectrum of the Laplace operator and its relation with the geometry of the underlying Riemannian manifold. Naturally, I have also included several smaller additions and minor corrections (for which I am grateful to several readers). Moreover, the organization of the chapters has been systematically rearranged.

Let me now briefly describe the contents:

In the first chapter, we introduce the basic geometric concepts of Riemannian geometry. We then begin the treatment of one of the fundamental objects and tools of Riemannian geometry, the so-called geodesics which are defined as locally shortest curves. Geodesics will reappear prominently in several later chapters. Here, we treat the existence of geodesics with two different methods, both of which are quite important in geometric analysis in general. Thus, the reader has the opportunity to understand the basic ideas of those methods in an elementary context before moving on to more difficult versions in subsequent chapters. The first method is based on the local existence and uniqueness of geodesics and will be applied again in Chapter 9 for two-dimensional harmonic maps. The second method is the heat flow method that gained prominence through Perelman's solution of the Poincaré conjecture by the Ricci flow method.

The second chapter introduces another fundamental concept, the one of a vector bundle. Besides the most basic one, the tangent bundle of a Riemannian manifold, many other vector bundles will appear in this book. The structure group of a vector bundle is a Lie group, and we shall therefore use this opportunity to also discuss Lie groups and their infinitesimal versions, the Lie algebras.

The third chapter then introduces basic concepts and methods from analysis. In particular, the Laplace-Beltrami operator is a fundamental object in Riemannian geometry. We show the essential properties of its spectrum and discuss relationships with the underlying geometry. We then turn to the operation of the Laplace operator on differential forms. We introduce de Rham cohomology groups and the essential tools from elliptic PDE for treating these groups. We prove the existence of harmonic forms representing cohomology classes both by a variational method, thereby introducing another of the basic schemes of geometric analysis, and by the heat flow method. The linear setting of cohomology classes allows us to understand some key ideas without the technical difficulties of nonlinear problems. We also discuss the spectrum of the Laplacian on differential forms. The important observation that the spectra for forms of different degrees are systematically related I learned from Johannes Rauh, whom I should like to thank for this.

The fourth chapter begins with fundamental geometric concepts. It treats the general theory of connections and curvature. We also introduce important functionals like the Yang-Mills functional and its properties, as well as minimal submanifolds. The Bochner method is applied to the first eigenvalue of the Laplacian and harmonic 1-forms on manifolds of positive Ricci curvature, as an example of the interplay between geometry and analysis. We also describe the method of Li and Yau for obtaining eigenvalue estimates through gradient bounds for eigenfunctions.

In the fifth chapter, we introduce Jacobi fields, prove the Rauch comparison theorems for Jacobi fields and apply these results to geodesics. We also develop the global geometry of spaces of nonpositive curvature.

These first five chapters treat the more elementary and basic aspects of the subject. Their results will be used in the remaining, more advanced chapters.

The sixth chapter treats Kähler manifolds and symmetric spaces as important examples of Riemannian manifolds in detail.

The seventh chapter is devoted to Morse theory and Floer homology.

In the eighth chapter, we treat harmonic maps between Riemannian manifolds. We prove several existence theorems and apply them to Riemannian geometry. The treatment uses an abstract approach based on convexity that should bring out the fundamental structures. We also display a representative sample of techniques from geometric analysis.

In the ninth chapter, we treat harmonic maps from Riemann surfaces. We encounter here the phenomenon of conformal invariance which makes this two-dimensional case distinctively different from the higher dimensional one.

Riemannian geometry has become the mathematical language of theoretical physics, whereas the rigorous demonstration of many results in theoretical physics requires deep tools from nonlinear analysis. Therefore, the tenth chapter explores some connections between physics, geometry and analysis. It treats variational problems from quantum field theory, in particular the Ginzburg–Landau and Seiberg–Witten equations, and a mathematical version of the nonlinear supersymmetric sigma model. In mathematical terms, the two-dimensional harmonic map problem is coupled with a Dirac field. The background material on spin geometry and Dirac operators is already developed in earlier chapters. The connections between geometry and physics are developed in more generality in my monograph [164].

A guiding principle for this textbook was that the material in the main body should be self-contained. The essential exception is that we use material about Sobolev spaces and linear elliptic and parabolic PDEs without giving proofs. This material is collected in Appendix A. Appendix B collects some elementary topological results about fundamental groups and covering spaces.

Also, in certain places in Chapter 7, we do not present all technical details, but rather explain some points in a more informal manner, in order to keep the size of that chapter within reasonable limits and not to lose the patience of the readers.

We employ both coordinate-free intrinsic notations and tensor notations depending on local coordinates. We usually develop a concept in both notations while we sometimes alternate in the proofs. Besides the fact that i am not a methodological purist, reasons for often preferring the tensor calculus to the more elegant and concise intrinsic one are the following. For the analytic aspects, one often has to employ results about (elliptic) partial differential equations (PDEs), and in order to check that the relevant assumptions like ellipticity hold and in order to make contact with the notations usually employed in PDE theory, one has to write down the differential equation in local coordinates. Also, manifold and important connections have been established between theoretical physics and our subject. In the physical literature, usually the tensor notation is employed, and therefore, familiarity with that notation is necessary for exploring those connections that have been found to be stimulating for the development of mathematics, or promise to be so in the future.

As appendices to most of the sections, we have written paragraphs with the title "Perspectives". The aim of those paragraphs is to place the material in a broader context and explain further results and directions without detailed proofs. The material of these Perspectives will not be used in the main body of the text. Similarly, after Chapter 5, we have inserted a section entitled "A short survey on curvature and topology" that presents an account of many global results of Riemannian geometry not covered in the main text. At the end of each chapter, some exercises for the reader are given. We trust the reader to be of sufficient perspicacity to understand our system of numbering and cross-references without further explanation.

I thank Miroslav Bačak and the copy editor for valuable corrections. I am grateful to the European Research Council for supporting my work with the Advanced Grant FP7-267087.

The development of the mathematical subject of Geometric Analysis, namely the investigation of analytical questions arising from a geometric context and in turn the application of analytical techniques to geometric problems, is to a large extent due to the work and the influence of Shing-Tung Yau. This book, like its previous editions, is dedicated to him.

Jürgen Jost

Contents

Chapter 1

Riemannian Manifolds

1.1 Manifolds and Differentiable Manifolds

A *topological space* is a set M together with a family $\mathcal{O}$ of subsets of M satisfying the following properties:

(i) $\Omega_1, \Omega_2 \in \mathcal{O} \Rightarrow \Omega_1 \cap \Omega_2 \in \mathcal{O}$,

(ii) for any index set $A : (\Omega_\alpha)_{\alpha \in A} \subset \mathcal{O} \Rightarrow \bigcup_{\alpha \in A} \Omega_\alpha \in \mathcal{O}$,

(iii) $\emptyset, M \in \mathcal{O}$.

The sets from $\mathcal{O}$ are called *open*. A topological space is called *Hausdorff* if for any two distinct points $p_1, p_2 \in M$ there exist open sets $\Omega_1, \Omega_2 \in \mathcal{O}$ with $p_1 \in \Omega_1, p_2 \in \Omega_2, \Omega_1 \cap \Omega_2 = \emptyset$. A covering $(\Omega_\alpha)_{\alpha \in A}$ (A an arbitrary index set) is called *locally finite* if each $p \in M$ has a neighborhood that intersects only finitely many Ω_α. M is called *paracompact* if any open covering possesses a locally finite refinement. This means that for any open covering $(\Omega_\alpha)_{\alpha \in A}$ there exists a locally finite open covering $(\Omega'_\beta)_{\beta \in B}$ with

$$\forall \beta \in B \; \exists \alpha \in A : \Omega'_\beta \subset \Omega_\alpha.$$

The condition of paracompactness ensures the existence of an important technical tool, the so-called partition of unity, see Lemma 1.1.1 below.

A map between topological spaces is called *continuous* if the preimage of any open set is again open. A bijective map which is continuous in both directions is called a *homeomorphism*.

J. Jost, *Riemannian Geometry and Geometric Analysis*, Universitext,
DOI 10.1007/978-3-642-21298-7_1, © Springer-Verlag Berlin Heidelberg 2011

Definition 1.1.1. A *manifold* M of *dimension* d is a connected paracompact Hausdorff space for which every point has a neighborhood U that is homeomorphic to an open subset Ω of $\mathbb{R}^d$. Such a homeomorphism

$$x : U \to \Omega$$

is called a *(coordinate) chart.*

An *atlas* is a family $\{U_\alpha, x_\alpha\}$ of charts for which the U_α constitute an open covering of M.

Remarks.

1. A point $p \in U_\alpha$ is determined by $x_\alpha(p)$; hence it is often identified with $x_\alpha(p)$. Often, also the index α is omitted, and the components of $x(p) \in \mathbb{R}^d$ are called *local coordinates* of p.

2. It is customary to write the Euclidean coordinates of $\mathbb{R}^d$ as

$$x = (x^1, \dots, x^d), \tag{1.1.1}$$

and these then are considered as local coordinates on our manifold M when $x : U \to \Omega$ is a chart.

As we shall see, local coordinates yield a systematic method for locally representing a manifold in such a manner that computations can be carried out. We shall now describe a concept that will allow us to utilize the framework of linear algebra for local computations as will be explored in §1.2 and beyond.

Definition 1.1.2. An atlas $\{U_\alpha, x_\alpha\}$ on a manifold is called *differentiable* if all chart transitions

$$x_\beta \circ x_\alpha^{-1} : x_\alpha(U_\alpha \cap U_\beta) \to x_\beta(U_\alpha \cap U_\beta)$$

are differentiable of class C^∞ (in case $U_\alpha \cap U_\beta \neq \emptyset$). A maximal differentiable atlas is called a differentiable structure, and a *differentiable manifold* of dimension d is a manifold of dimension d with a differentiable structure. From now on, all atlases are supposed to be differentiable. Two atlases are called compatible if their union is again an atlas. In general, a chart is called compatible with an atlas if adding the chart to the atlas yields again an atlas. An atlas is called maximal if any chart compatible with it is already contained in it.

Remarks.

1. One could also require a weaker differentiability property than C^∞, for instance C^k, i.e., that all chart transitions be k times continuously differentiable, for some $k \in \mathbb{N}$. C^∞ is convenient as one never needs to worry about the order of differentiability. The spaces C^k for $k \in \mathbb{N}$, on the other hand, offer the advantage of being Banach spaces.