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A Course in Topological Combinatorics

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For L., with whom I fell, first from the bicycle and then in love

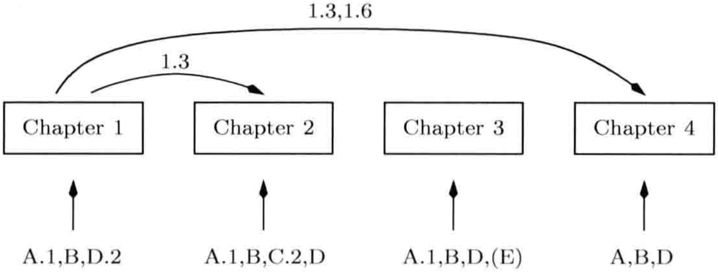
Preface

Topological combinatorics is a very young and exciting field of research in mathematics. It is mostly concerned with the application of the many powerful tools of algebraic topology to combinatorial problems. One of its early landmarks was Lovász's proof of the Kneser conjecture published in 1978. The combination of the two mathematical fields—topology and combinatorics—has led to many surprising and elegant proofs and results.

In this textbook I present some of the most beautiful and accessible results from topological combinatorics. It grew out of several courses that I have taught at Freie Universität Berlin, and is based on my personal taste and what I believe is suitable for the classroom. In particular, it aims for a clear and vivid presentation rather than encyclopedic completeness.

The text is designed for an advanced undergraduate level. Primarily it serves as a basis for a course, but is written in such a way that it just as well may be read by students independently. The textbook is essentially self-contained. Only some basic mathematical experience and knowledge—in particular some linear algebra—is required. An extensive appendix allows the instructor to design courses for students with very different prerequisites. Some of those designs will be sketched later on.

The textbook has four main chapters and several appendices. Each chapter ends with an accompanying and complementing set of exercises. The main chapters are mostly independent of each other and thus allow considerable flexibility for an individual course design. The dependencies are roughly as follows.



Suggested Course Outlines

For students with previous knowledge of graph theory and the basics of algebraic topology including simplicial homology theory. Use Chaps. 1–4. Whenever concepts and results on partially ordered sets and their topology from Appendix C or on group actions from Appendix D are missing, they should be included. Oliver's Theorem 3.17, which is proven in Appendix E, can easily be used as a black box. If the students are experienced with homology and if time permits, I recommend studying Appendix E after Chap. 3.

For students with previous knowledge of the basics of algebraic topology including simplicial homology theory only. Proceed as in the last case and provide the basics of graph theory from Appendix A along the way.

For students with previous knowledge of graph theory only. I recommend that the instructor introduces some basic topology with Sects. B.1 and B.3, and then presents Chap. 1, skipping the homological proofs. Before Sect. 1.6 I recommend giving a topology crash course with Sects. B.4–B.9. Proceed with Chaps. 2–4 and add concepts and results from Appendices C and D as needed. Apply Theorem 3.17 as a black box and use Appendix E as a motivation to convince students to study algebraic topology.

For motivated students with neither graph theory nor algebraic topology knowledge. Proceed as in the last case and provide the basics from graph theory from Appendix A along the way.

Acknowledgments

First of all, I would like to thank all the authors of research papers and textbooks—several of them I know personally—on which this book is based. I am thankful to Martin Aigner for helpful advice and for supporting the initial idea of the project, and to Günter M. Ziegler for providing excellent working conditions in his research group at Technische Universität Berlin. I am indebted to all the students who took part in my courses on the subject, and to all of the colleagues who helped me with discussions, suggestions, and proofreading. In particular, I want to thank Anna Gundert, Nicolina Hauke, Daria Schymura, Felix Breuer, Aaron Dall, Anton Dochtermann, Frederik von Heymann, Frank Lutz, Benjamin Matschke, Marc Pfetsch, and Carsten Schultz. I also want to thank David Kramer from Springer New York for his very helpful copy editing, and finally, Hans Koelsch and Kaitlin Leach from Springer New York for their competent support of this project.

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List of Symbols and Typical Notation

$[n] = \{1, 2, \ldots, n\}$ the set of natural numbers from 1 to n

$|S|$ the number of elements of a set S

$\lfloor x \rfloor$ the largest integer less than or equal to x

$k \mid n$ notation for "k divides n"

$\subseteq$ the subset relation

$\subset$ the proper subset relation

$S = S_1 \dot{\cup} \cdots \dot{\cup} S_n$ a partition of the set S, i.e., $S = S_1 \cup \cdots \cup S_n$ and $S_i \cap S_j = \emptyset$ for all $i \neq j$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ the number of k-element subsets of an n-element set

$\binom{X}{k} = \{S \subseteq X : |S| = k\}$ the set of k-element subsets of a set X

$X + Y = X \times \{0\} \cup Y \times \{1\}$ the sum of sets X and Y

K, L abstract simplicial complexes

Δ, Γ geometric simplicial complexes

$\tau \leq \sigma$ notation for "the simplex τ is a face of the simplex σ"

$\sigma^n = \text{conv}(\{e_1, \ldots, e_{n+1}\})$ the standard geometric n-simplex

Δ^n the geometric simplicial complex given by σ^n and all its faces

$K(\Delta)$ the abstract simplicial complex associated with the geometric complex Δ (cf. page 176)

$|\Delta|$ the polyhedron of the geometric simplicial complex Δ

$|K|$ a geometric realization of the abstract complex K or its polyhedron

$\mathcal{P}(X)$ the power set of X, i.e., $\mathcal{P}(X) = \{A : A \subseteq X\}$

2^X will be identified with the power set of X

$2^{[n]}$ will be identified with the power set of $[n]$

2^σ will be identified with the power set of σ, in this notation refers to the abstract simplicial complex given by the simplex σ and all its faces

$\|\cdot\| = \|\cdot\|_2$ the Euclidean norm

$\|\cdot\|_\infty$ the maximum norm

$\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ the n-dimensional unit ball

$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ the $(n-1)$-dimensional unit sphere

$Q^{n+1} = \operatorname{conv}(\{\pm e_1, \ldots, \pm e_{n+1}\})$ the $(n+1)$-dimensional cross polytope

Γ^n the n-dimensional geometric simplicial complex associated with the boundary of the cross polytope Q^{n+1}

Contents

Chapter 1
Fair-Division Problems

Almost every day, we encounter fair-division problems: in the guise of dividing a piece of cake, slicing a ham sandwich, or by dividing our time with respect to the needs and expectations of family, friends, work, etc.

The mathematics of such fair-division problems will serve us as a first representative example for the interplay between combinatorics and topology.

In this chapter we will consider two important concepts: *envy-free fair division* and *consensus division*. These concepts lead to different topological tools that we may apply. On the one hand, there is Brouwer's fixed-point theorem, and on the other hand, there is the theorem of Borsuk and Ulam. These topological results surprisingly turn out to have combinatorial analogues: the lemmas of Sperner and Tucker. Very similar in nature, they guarantee a simplex with a certain labeling in a labeled simplicial complex.

The chapter is organized in such a way that we will discuss in turn a topological result, its combinatorial analogue, and the corresponding fair-division problem.

1.1 Brouwer's Fixed-Point Theorem and Sperner's Lemma

Brouwer's fixed-point theorem states that any continuous map from a ball of any dimension to itself has a fixed point. In two dimensions this can be illustrated as follows. Take two identical maps of Berlin or any other ball-shaped city. Now crumple one of the maps as you like and throw it on the other, flat, map as shown in Fig. 1.1. Then there exists a location in the city that on the crumpled map is exactly above the same place on the flat map.

For the general formulation of Brouwer's theorem, recall that the n-dimensional Euclidean ball is given by all points of distance at most 1 from the origin in n-dimensional Euclidean space, i.e.,

$$\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

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Fig. 1.1 A city map twice

Theorem 1.1 (Brouwer). *Every continuous map $f : \mathbb{B}^n \to \mathbb{B}^n$ from the n-dimensional ball $\mathbb{B}^n$ to itself has a fixed point, i.e., there exists an $x \in \mathbb{B}^n$ such that $f(x) = x$.*

The first proof that we provide for this theorem relies on a beautiful combinatorial lemma that we will discuss in the next section. There also exists a very short and simple proof using homology theory that we present on page 6.

Sperner's Lemma

Brouwer's fixed-point theorem is intimately related to a combinatorial lemma by Sperner that deals with labelings of triangulations of the simplex. Consider the standard n-simplex given as the convex hull of the standard basis vectors $\{e_1, \dots, e_{n+1}\} \subseteq \mathbb{R}^{n+1}$, see Fig. 1.2:

$$\begin{aligned}\sigma^n &= \operatorname{conv}(\{e_1, \dots, e_{n+1}\}) \\ &= \left\{ t_1 e_1 + \cdots + t_{n+1} e_{n+1} : t_i \geq 0, \sum_{i=1}^{n+1} t_i = 1 \right\} \\ &= \left\{ (t_1, \dots, t_{n+1}) : t_i \geq 0, \sum_{i=1}^{n+1} t_i = 1 \right\}.\end{aligned}$$

By Δ^n we denote the (geometric) simplicial complex given by σ^n and all its faces, i.e., $\Delta^n = \{\tau : \tau \leq \sigma^n\}$. Assume that K is a subdivision of Δ^n. We may think of K as being obtained from Δ^n by adding extra vertices. For precise definitions and more details on simplicial complexes we refer to Appendix B. For any n, denote the set $\{1, \dots, n\}$ by $[n]$. In the definition of a Sperner labeling we will use labels from 1 to $n + 1$, i.e., labels from the set $[n + 1]$.

Definition 1.2. A *Sperner labeling* is a labeling $\lambda : \operatorname{vert}(K) \to [n + 1]$ of the vertices of K satisfying

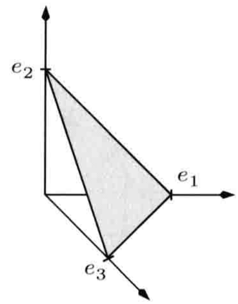

Fig. 1.2 The standard 2-simplex $\sigma^2 = \text{conv}(\{e_1, e_2, e_3\})$

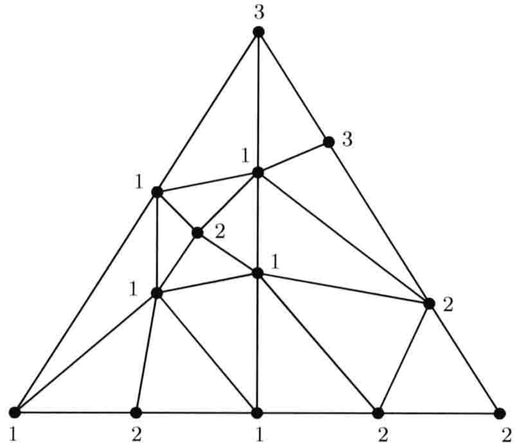

Fig. 1.3 A Sperner-labeled triangulation of a 2-simplex

$$\lambda(v) \in \{i \in [n+1] : v_i \neq 0\}$$

for all $v \in \text{vert}(K)$.

More intuitively, a Sperner labeling is the following. Consider the minimal face of Δ^n that contains v. Say it is given by the convex hull of $e_{i_1}, \ldots, e_{i_k}$. Then v is allowed to get labels only from $\{i_1, \ldots, i_k\}$. In particular, the vertices e_i obtain label i, while a vertex along the edge spanned by e_i and e_j obtains the label i or j, and so on. For an illustration see Fig. 1.3.

Call an n-simplex of K *fully labeled* (with respect to λ) if its $n+1$ vertices obtain distinct labels, i.e., if all possible labels from the set $[n+1]$ appear.

Lemma 1.3 (Sperner [Spe28]). *Let $\lambda : \text{vert}(K) \to [n+1]$ be a Sperner labeling of a triangulation K of the n-dimensional simplex. Then there exists a fully labeled n-simplex in K. More precisely, the number of fully labeled n-simplices is odd.*

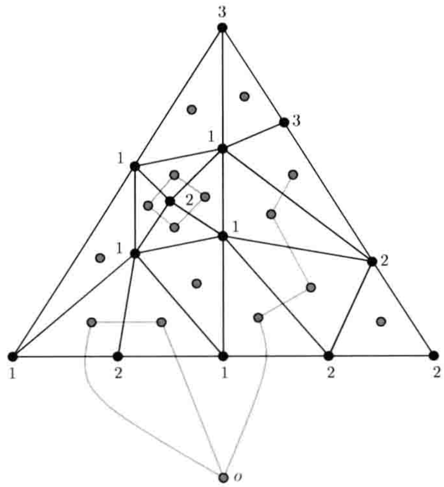

Fig. 1.4 The graph associated to a Sperner labeling

We will now present two inductive proofs of this amazing lemma. The first is a combinatorial construction that constructs one of the desired simplices, while the other is algebraic and uses the concept of a chain complex of a simplicial complex. The inductive proofs reveal a typical phenomenon: while we are mainly interested in the existence of a fully labeled simplex, the induction works only for the stronger statement that there is an odd number of fully labeled simplices.

A third proof, given on page 7 of this section, proves the Sperner lemma as an application of Brouwer's fixed-point theorem.

Proof (combinatorial). The lemma is clearly valid for $n = 1$. Now let $n \geq 2$ and consider the $(n-1)$-dimensional face τ of Δ^n given by the convex hull of $e_1, \dots, e_n$. Note that K restricted to τ is Sperner labeled with label set $[n]$. We construct a graph as follows. Let the vertex set be all n-simplices of K plus one extra vertex o. The extra vertex o is connected by an edge to all n-simplices that have an $(n-1)$-simplex as a face that is labeled with all labels of $[n]$ and lies within τ. Two n-simplices are connected by an edge if and only if they share an $(n-1)$-dimensional face labeled with all of $[n]$. See Fig. 1.4 for an example of the resulting graph.

By the induction hypothesis, the vertex o has odd degree, since there is an odd number of fully labeled simplices in the labeling restricted to τ. All the other vertices have degree zero, one, or two. To see this, consider the set of labels an n-simplex obtains: either it does not contain $[n]$, it is $[n+1]$, or it is $[n]$. In the first case, the simplex has degree zero; in the second, it has degree one; and in the last case, it has degree two, since exactly two faces obtain all of $[n]$ as label set; compare Fig. 1.5.

Hence the vertices of degree one other than o (which may have degree one) correspond to the fully labeled simplices. Now, the number of vertices of odd degree