

S.M. NIKOLSKY

**A Course
of Mathematical
Analysis**

Volume

2

MIR PUBLISHERS MOSCOW

S. M. NIKOLSKY

Member, USSR Academy of Sciences

A Course of Mathematical Analysis

Volume 2

Translated from the Russian

by

V. M. VOLOSOV, D. Sc.



MIR PUBLISHERS

MOSCOW

Contents

Chapter 12. Multiple Integrals	9
§ 12.1. Introduction	9
§ 12.2. Jordan Squarable Sets	11
§ 12.3. Some Important Examples of Squarable Sets	17
§ 12.4. One More Test for Measurability of a Set. Area in Polar Coordinates	19
§ 12.5. Jordan Measurable Three-dimensional and n -dimensional Sets	20
§ 12.6. The Notion of Multiple Integral	24
§ 12.7. Upper and Lower Integral Sums. Key Theorem	27
§ 12.8. Integrability of a Continuous Function on a Measurable Closed Set. Some Other Integrability Conditions	32
§ 12.9. Set of Lebesgue Measure Zero	34
§ 12.10. Proof of Lebesgue's Theorem. Connection Between Integrability and Boundedness of a Function	35
§ 12.11. Properties of Multiple Integrals	38
§ 12.12. Reduction of Multiple Integral to Iterated Integral	41
§ 12.13. Continuity of Integral Dependent on Parameter	48
§ 12.14. Geometrical Interpretation of the Sign of a Determinant	51
§ 12.15. Change of Variables in Multiple Integral. Simplest Case	54
§ 12.16. Change of Variables in Multiple Integral. General Case	56
§ 12.17. Proof of Lemma 1, § 12.16	59
§ 12.18. Double Integral in Polar Coordinates	63
§ 12.19. Triple Integral in Spherical Coordinates	65
§ 12.20. General Properties of Continuous Operators	67
§ 12.21. More on Change of Variables in Multiple Integral	68
§ 12.22. Improper Integral with Singularities on the Boundary of the Domain of Integration. Change of Variables	71
§ 12.23. Surface Area	73
Chapter 13. Scalar and Vector Fields. Differentiation and Integration of Integral with Respect to Parameter. Improper Integrals	80
§ 13.1. Line Integral of the First Type	80
§ 13.2. Line Integral of the Second Type	81
§ 13.3. Potential of a Vector Field	83
§ 13.4. Orientation of a Domain in the Plane	91
§ 13.5. Green's Formula. Computing Area with the Aid of Line Integral ..	92
§ 13.6. Surface Integral of the First Type	96
§ 13.7. Orientation of a Surface	98
§ 13.8. Integral over an Oriented Domain in the Plane	102
§ 13.9. Flux of a Vector Through an Oriented Surface	104
§ 13.10. Divergence. Gauss-Ostrogradsky Theorem	107
§ 13.11. Rotation of a Vector. Stokes' Theorem	114

§ 13.12.	Differentiation of Integral with Respect to Parameter	118
§ 13.13.	Improper Integrals	121
§ 13.14.	Uniform Convergence of Improper Integrals	128
§ 13.15.	Uniformly Convergent Integral over Unbounded Domain	135
§ 13.16.	Uniformly Convergent Improper Integral with Variable Singularity ..	140
Chapter 14.	Normed Linear Spaces. Orthogonal Systems ..	147
§ 14.1.	Space C of Continuous Functions	147
§ 14.2.	Spaces L^1 , L_p and l_p	149
§ 14.3.	Spaces L_2 and l_2	154
§ 14.4.	Approximation with Finite Functions	156
§ 14.5.	Linear Spaces. Fundamentals of the Theory of Normed Linear Spaces	163
§ 14.6.	Orthogonal Systems in Space with Scalar Product	170
§ 14.7.	Orthogonalization Process	181
§ 14.8.	Properties of Spaces $L_2(\Omega)$ and $l_2(\Omega)$	185
§ 14.9.	Complete Systems of Functions in the Spaces C , L_2 and L^1 (l_2 , l)	187
Chapter 15.	Fourier Series. Approximation of Functions with Polynomials	188
§ 15.1.	Preliminaries	188
§ 15.2.	Dirichlet's Sum	195
§ 15.3.	Formulas for the Remainder of Fourier's Series	197
§ 15.4.	Oscillation Lemmas	199
§ 15.5.	Test for Convergence of Fourier Series. Completeness of Trigonometric System of Functions	203
§ 15.6.	Complex Form of Fourier Series	211
§ 15.7.	Differentiation and Integration of Fourier Series	213
§ 15.8.	Estimating the Remainder of Fourier's Series	216
§ 15.9.	Gibbs' Phenomenon	217
§ 15.10.	Fejér's Sums	221
§ 15.11.	Elements of the Theory of Fourier Series for Functions of Several Variables	225
§ 15.12.	Algebraic Polynomials. Chebyshev's Polynomials	235
§ 15.13.	Weierstrass' Theorem	236
§ 15.14.	Legendre's Polynomials	237
Chapter 16.	Fourier Integral. Generalized Functions	240
§ 16.1.	Notion of Fourier Integral	240
§ 16.2.	Lemma on Change of Order of Integration	243
§ 16.3.	Convergence of Fourier's Single Integral	245
§ 16.4.	Fourier Transform and Its Inverse. Iterated Fourier Integral. Fourier Cosine and Sine Transforms	247
§ 16.5.	Differentiation and Fourier Transformation	249
§ 16.6.	Space S	250
§ 16.7.	Space S' of Generalized Functions	255
§ 16.8.	Many-dimensional Fourier Integrals and Generalized Functions	265
§ 16.9.	Finite Step Functions. Approximation in the Mean Square	273
§ 16.10.	Plancherel's Theorem. Estimating Speed of Convergence of Fourier's Integral	278
§ 16.11.	Generalized Periodic Functions	283
Chapter 17.	Differentiable Manifolds and Differential Forms	289
§ 17.1.	Differentiable Manifolds	289
§ 17.2.	Boundary of a Differentiable Manifold and Its Orientation	299
§ 17.3.	Differential Forms	310
§ 17.4.	Stokes' Theorem	220

Chapter 18. Supplementary Topics	326
§ 18.1. Generalized Minkowski's Inequality	326
§ 18.2. Sobolev's Regularization of Function	329
§ 18.3. Convolution	333
§ 18.4. Partition of Unity	335
Chapter 19. Lebesgue Integral	338
§ 19.1. Lebesgue Measure	338
§ 19.2. Measurable Functions	348
§ 19.3. Lebesgue Integral	355
§ 19.4. Lebesgue Integral on Unbounded Set	388
§ 19.5. Sobolev's Generalized Derivative	392
§ 19.6. Space D' of Generalized Functions	404
§ 19.7. Incompleteness of Space L'_p	407
§ 19.8. Generalization of Jordan Measure	408
§ 19.9. Riemann-Stieltjes Integral	414
§ 19.10. Stieltjes Integral	415
§ 19.11. Generalization of Lebesgue Integral	423
§ 19.12. Lebesgue-Stieltjes Integral	424
§ 19.13. Extension of Functions. Weierstrass' Theorem	433
Name Index	437
Subject Index	438

S.M. NIKOLSKY

**A Course
of Mathematical
Analysis**

Volume

2

MIR PUBLISHERS MOSCOW



S. M. NIKOLSKY

Member, USSR Academy of Sciences

A Course of Mathematical Analysis

Volume 2

Translated from the Russian

by

V. M. VOLOSOV, D. Sc.



MIR PUBLISHERS

MOSCOW

First published 1977
Second printing 1981
Third printing 1985
Fourth printing 1987

TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

Our address is: USSR, 129820, Moscow I-110, GSP

Pervy Rizhsky Pereulok, 2

MIR PUBLISHERS

На английском языке

© Издательство «Наука», 1975

© English translation, Mir Publishers, 1977

Contents

Chapter 12. Multiple Integrals	9
§ 12.1. Introduction	9
§ 12.2. Jordan Squarable Sets	11
§ 12.3. Some Important Examples of Squarable Sets	17
§ 12.4. One More Test for Measurability of a Set. Area in Polar Coordinates	19
§ 12.5. Jordan Measurable Three-dimensional and n -dimensional Sets	20
§ 12.6. The Notion of Multiple Integral	24
§ 12.7. Upper and Lower Integral Sums. Key Theorem	27
§ 12.8. Integrability of a Continuous Function on a Measurable Closed Set. Some Other Integrability Conditions	32
§ 12.9. Set of Lebesgue Measure Zero	34
§ 12.10. Proof of Lebesgue's Theorem. Connection Between Integrability and Boundedness of a Function	35
§ 12.11. Properties of Multiple Integrals	38
§ 12.12. Reduction of Multiple Integral to Iterated Integral	41
§ 12.13. Continuity of Integral Dependent on Parameter	48
§ 12.14. Geometrical Interpretation of the Sign of a Determinant	51
§ 12.15. Change of Variables in Multiple Integral. Simplest Case	54
§ 12.16. Change of Variables in Multiple Integral. General Case	56
§ 12.17. Proof of Lemma 1, § 12.16	59
§ 12.18. Double Integral in Polar Coordinates	63
§ 12.19. Triple Integral in Spherical Coordinates	65
§ 12.20. General Properties of Continuous Operators	67
§ 12.21. More on Change of Variables in Multiple Integral	68
§ 12.22. Improper Integral with Singularities on the Boundary of the Domain of Integration. Change of Variables	71
§ 12.23. Surface Area	73
Chapter 13. Scalar and Vector Fields. Differentiation and Integration of Integral with Respect to Parameter. Improper Integrals	80
§ 13.1. Line Integral of the First Type	80
§ 13.2. Line Integral of the Second Type	81
§ 13.3. Potential of a Vector Field	83
§ 13.4. Orientation of a Domain in the Plane	91
§ 13.5. Green's Formula. Computing Area with the Aid of Line Integral ..	92
§ 13.6. Surface Integral of the First Type	96
§ 13.7. Orientation of a Surface	98
§ 13.8. Integral over an Oriented Domain in the Plane	102
§ 13.9. Flux of a Vector Through an Oriented Surface	104
§ 13.10. Divergence. Gauss-Ostrogradsky Theorem	107
§ 13.11. Rotation of a Vector. Stokes' Theorem	114

§ 13.12. Differentiation of Integral with Respect to Parameter	118
§ 13.13. Improper Integrals	121
§ 13.14. Uniform Convergence of Improper Integrals	128
§ 13.15. Uniformly Convergent Integral over Unbounded Domain	135
§ 13.16. Uniformly Convergent Improper Integral with Variable Singularity ..	140
Chapter 14. Normed Linear Spaces. Orthogonal Systems ..	147
§ 14.1. Space C of Continuous Functions	147
§ 14.2. Spaces L^1 , L_p and l_p	149
§ 14.3. Spaces L_2 and l_2	154
§ 14.4. Approximation with Finite Functions	156
§ 14.5. Linear Spaces. Fundamentals of the Theory of Normed Linear Spaces	163
§ 14.6. Orthogonal Systems in Space with Scalar Product	170
§ 14.7. Orthogonalization Process	181
§ 14.8. Properties of Spaces $L_2(\Omega)$ and $l_2(\Omega)$	185
§ 14.9. Complete Systems of Functions in the Spaces C , L_2 and L^1 (l_2 , l)	187
Chapter 15. Fourier Series. Approximation of Functions with Polynomials	188
§ 15.1. Preliminaries	188
§ 15.2. Dirichlet's Sum	195
§ 15.3. Formulas for the Remainder of Fourier's Series	197
§ 15.4. Oscillation Lemmas	199
§ 15.5. Test for Convergence of Fourier Series. Completeness of Trigonometric System of Functions	203
§ 15.6. Complex Form of Fourier Series	211
§ 15.7. Differentiation and Integration of Fourier Series	213
§ 15.8. Estimating the Remainder of Fourier's Series	216
§ 15.9. Gibbs' Phenomenon	217
§ 15.10. Fejér's Sums	221
§ 15.11. Elements of the Theory of Fourier Series for Functions of Several Variables	225
§ 15.12. Algebraic Polynomials. Chebyshev's Polynomials	235
§ 15.13. Weierstrass' Theorem	236
§ 15.14. Legendre's Polynomials	237
Chapter 16. Fourier Integral. Generalized Functions	240
§ 16.1. Notion of Fourier Integral	240
§ 16.2. Lemma on Change of Order of Integration	243
§ 16.3. Convergence of Fourier's Single Integral	245
§ 16.4. Fourier Transform and Its Inverse. Iterated Fourier Integral. Fourier Cosine and Sine Transforms	247
§ 16.5. Differentiation and Fourier Transformation	249
§ 16.6. Space S	250
§ 16.7. Space S' of Generalized Functions	255
§ 16.8. Many-dimensional Fourier Integrals and Generalized Functions	265
§ 16.9. Finite Step Functions. Approximation in the Mean Square	273
§ 16.10. Plancherel's Theorem. Estimating Speed of Convergence of Fourier's Integral	278
§ 16.11. Generalized Periodic Functions	283
Chapter 17. Differentiable Manifolds and Differential Forms	289
§ 17.1. Differentiable Manifolds	289
§ 17.2. Boundary of a Differentiable Manifold and Its Orientation	299
§ 17.3. Differential Forms	310
§ 17.4. Stokes' Theorem	220

Chapter 18. Supplementary Topics	326
§ 18.1. Generalized Minkowski's Inequality	326
§ 18.2. Sobolev's Regularization of Function	329
§ 18.3. Convolution	333
§ 18.4. Partition of Unity	335
Chapter 19. Lebesgue Integral	338
§ 19.1. Lebesgue Measure	338
§ 19.2. Measurable Functions	348
§ 19.3. Lebesgue Integral	355
§ 19.4. Lebesgue Integral on Unbounded Set	388
§ 19.5. Sobolev's Generalized Derivative	392
§ 19.6. Space D' of Generalized Functions	404
§ 19.7. Incompleteness of Space L'_p	407
§ 19.8. Generalization of Jordan Measure	408
§ 19.9. Riemann-Stieltjes Integral	414
§ 19.10. Stieltjes Integral	415
§ 19.11. Generalization of Lebesgue Integral	423
§ 19.12. Lebesgue-Stieltjes Integral	424
§ 19.13. Extension of Functions. Weierstrass' Theorem	433
Name Index	437
Subject Index	438

The major part of this two-volume textbook stems from the course in mathematical analysis given by the author for many years at the Moscow Physico-technical Institute.

The first volume consisting of eleven chapters includes an introduction (Chapter 1) which treats of fundamental notions of mathematical analysis using an intuitive concept of a limit. With the aid of visual interpretation and some considerations of a physical character it establishes the relationship between the derivative and the integral and gives some elements of differentiation and integration techniques necessary to those readers who are simultaneously studying physics.

The notion of a real number is interpreted in the first volume (Chapter 2) on the basis of its representation as an infinite decimal.

Chapters 3-11 contain the following topics: Limit of Sequence, Limit of Function, Functions of One Variable, Functions of Several Variables, Indefinite Integral, Definite Integral, Some Applications of Integrals, Series.

Multiple Integrals

§ 12.1. Introduction

Let us consider a continuous surface, lying in the three-dimensional space with rectangular coordinates (x, y, z) , which is determined by an equation

$$z = f(Q) = f(x, y) \quad (Q = (x, y) \in \Omega)$$

where Ω is a bounded (two-dimensional) set possessing area (two-dimensional measure*). For instance, Ω can be a circle, a rectangle, an ellipse, etc. We shall suppose that the function $f(x, y)$ is positive. Let us state the following problem: it is required to find the volume of the solid bounded above by the given surface and below by the plane $z = 0$, its lateral boundary being the cylindrical surface with generators parallel to the z -axis and passing through the boundary curve γ of the set Ω .

To determine the sought-for volume we resort to the following natural procedure.

The set Ω is divided into a finite number N of parts (subdomains)

$$\Omega_1, \dots, \Omega_N \quad (1)$$

any two of which either do not intersect or intersect only along some parts of their boundaries. Let these subdomains be such that they possess areas (two-dimensional measures) which we shall denote as $m\Omega_1, \dots, m\Omega_N$ respectively.

Let us introduce the notion of the diameter of a set: if A is a set in the plane its *diameter* $d(A)$ is defined as

$$d(A) = \sup_{P', P'' \in A} |P' - P''|$$

where the supremum is taken over all the pairs of points P', P'' belonging to A .

Now we choose an arbitrary point $Q_j = (\xi_j, \eta_j)$ ($j = 1, \dots, N$) in each part Ω_j and form the sum

$$V_N = \sum_{j=1}^N f(Q_j) m\Omega_j \quad (2)$$

* See § 12.2.

which can be regarded as an approximation to the sought-for volume V . We can naturally suppose that the smaller the diameters $d(\Omega_j)$ of the subdomains Ω_j are, the higher is the accuracy of the approximation $V \approx V_N$. Therefore the volume V of the solid in question can be defined as the limit

$$V = \lim_{\max d(\Omega_j) \rightarrow 0} \sum_{j=1}^N f(Q_j) m\Omega_j \quad (3)$$

to which sum (2) tends when the maximum diameter of the subdomains of partitions (1) are made to tend to zero provided that this limit exists and is independent of the way in which the sequence of partitions (1) is chosen.

Now we can abstract from the problem of finding the volume of a solid and regard expression (3) as the result of an operation performed on the given function f defined in Ω . It is called the *Riemann double integral of the function f over the domain Ω* and is denoted

$$V = \lim_{\max d(\Omega_j) \rightarrow 0} \sum_{j=1}^N f(Q_j) m\Omega_j = \iint_{\Omega} f(x, y) dx dy = \int_{\Omega} f(Q) dQ = \int_{\Omega} f d\Omega$$

Let us consider a problem leading to the notion of the triple integral. Suppose that there is a physical body occupying a domain (set) Ω in the three-dimensional space with rectangular coordinates (x, y, z) and that the mass of the body is distributed (nonuniformly, in the general case) over Ω with volume density $\mu(x, y, z) = \mu(Q)$ ($Q = (x, y, z) \in \Omega$). It is required to determine the total mass of the body Ω .

To solve this problem it is natural to partition Ω into N parts $\Omega_1, \dots, \Omega_N$ whose volumes (three-dimensional measures) are $m\Omega_1, \dots, m\Omega_N$ (on condition that these volumes exist), to choose an arbitrary point $Q_j = (x_j, y_j, z_j) \in \Omega_j$ ($j = 1, \dots, N$) in each of the parts and to define the sought-for mass as the limit

$$M = \lim_{\max d(\Omega_j) \rightarrow 0} \sum_{j=1}^N \mu(Q_j) m\Omega_j \quad (4)$$

Expression (4) can again be regarded as the result of an operation performed on the function μ defined in the three-dimensional set Ω . It is called the *Riemann triple integral of f on Ω* and is denoted as

$$M = \lim_{\max d(\Omega_j) \rightarrow 0} \sum \mu(Q_j) m\Omega_j = \iiint_{\Omega} \mu(Q) dQ = \iiint_{\Omega} \mu(x, y, z) dx dy dz$$

The *Riemann n -fold multiple integral* is defined in the same way.

We shall see that the theory of (Riemann) multiple integration which includes existence theorems and theorems on the additive properties of the integral can be presented for the n -dimensional case in exactly the same manner as in the case of dimension 1. However, the theory of multiple integrals involves some specific difficulties which were not encountered in the theory of one-fold integration.

The matter is that the (Riemann) one-fold integral was defined for an extremely simple set, namely, for a closed interval $[a, b]$ which was partitioned into parts which were also closed intervals. Therefore we had no difficulties in defining the lengths (*one-dimensional measures*) of the intervals. But in the case of a double integral or, generally, n -fold integral, the domain of integration Ω can be split into parts with curvilinear boundaries, which makes it necessary to define the notion of the area or, generally, of the n -dimensional measure of such a part. A similar question would also appear in the case $n = 1$ if we defined the one-fold Riemann integral for a set of a more complex structure than that of a closed interval.

In this connection we must state a strict definition of the notion of measure of a set and investigate the properties of the measure. Therefore we begin this chapter with the theory of the Jordan* measure closely related to the theory of the Riemann integral. This theory forms the basis for the representation of the theory of the Riemann multiple integral. The latter theory provides an important method for evaluation of n -fold multiple integrals by reducing them to the so-called *iterated (repeated) integrals* involving n one-fold integrations with respect to each of the variables; in many important cases this procedure admits of the application of the Newton-Leibniz theorem established for one-fold integrals.

§ 12.2. Jordan Squarable Sets

Let us consider the plane $R = R_2$ with a definitely chosen rectangular coordinates (x, y) ; this coordinate system will also be denoted by the same letter R .

If some other coordinate system (ξ, η) is taken in the same plane we shall denote the plane (and the new coordinate system) by R' .

A rectangle Δ in the plane R will be regarded as the simplest set. It can be defined analytically by assuming that there is a system of rectangular coordinates R' in which Δ is representable as a set of points (ξ, η) satisfying inequalities of the form

$$a_1 \leq \xi \leq a_2, \quad b_1 \leq \eta \leq b_2 \quad (1)$$

where a_1, a_2, b_1 and b_2 are some numbers such that $a_1 < a_2$ and $b_1 < b_2$. The coordinate system R' possesses the property that the sides of Δ are parallel to its coordinate axes. To stress that the sides of Δ are parallel to the coordinate axes of the system R' we shall write $\Delta = \Delta_{R'}$. The rectangles of the type of Δ are understood here as closed sets (closed rectangles including their boundaries).

Now we define the notion of an *elementary figure* σ : a set $\sigma \subset R$ will be called an elementary figure if it is representable as a (set-theoretic) sum of a finite number of rectangles $\Delta \subset R$ any two of which either do not intersect or intersect only along some parts of their boundaries. The area $|\sigma|$ of

* C. Jordan (1838-1922), a French mathematician.

a two-dimensional elementary figure σ is defined as the sum of the areas of the rectangles Δ of which σ is composed.

A given figure σ can be represented as a finite sum (union) of rectangles Δ in infinitely many ways but the area $|\sigma|$ is independent of the representation. This assertion can readily be proved using the means of elementary geometry, and we do not dwell on it here.

An empty set is also regarded as a figure and its measure (area) is understood as being zero.

In inequalities (1) defining a rectangle Δ we assumed that $a_1 < a_2$ and $b_1 < b_2$. Therefore separate points and line segments will not be regarded as rectangles; our representation of the theory of measure will not involve such "degenerate" rectangles.

If an elementary figure σ is representable as a sum of rectangles Δ whose sides are parallel to the axes of the coordinate system R we shall write $\sigma = \sigma_R$.

Enumerated below are some simple properties of elementary figures σ . Their proofs are quite simple and we do not dwell on them here.

(a) If $\sigma_1 \subset \sigma_2$ then $|\sigma_1| \leq |\sigma_2|$.

(b) The (set-theoretic) sum of figures σ'_R and σ''_R is a figure σ_R and there holds the inequality

$$|\sigma'_R + \sigma''_R| \leq |\sigma'_R| + |\sigma''_R|$$

It becomes an equality if σ'_R and σ''_R either do not intersect each other or intersect only along some parts of their boundaries.

(c) The difference of two figures σ'_R and σ''_R is not necessarily a closed set and therefore it may not be an elementary figure. It can only be a figure (possibly empty) if $\sigma'_R \subset \sigma''_R$ or if σ'_R and σ''_R do not intersect. However, the closure $\overline{\sigma'_R - \sigma''_R}$ of this difference is always a figure and there holds the inequality

$$|\overline{\sigma'_R - \sigma''_R}| \geq |\sigma'_R| - |\sigma''_R|$$

It turns into an equality if $\sigma''_R \subset \sigma'_R$.

(d) If a figure σ_R is divided into two parts by a line parallel to one of the coordinate axes of the system R these parts are two figures σ'_R and σ''_R .

To these properties we shall add two more; one of them is connected with the notion of a network.

Let us take an arbitrary natural number N and construct two families of straight lines: $x = kh$ and $y = lh$ ($h = 2^{-N}$; $k, l = 0, \pm 1, \pm 2, \dots$). These families determine the rectangular network S_N dividing R into the squares Δ_h with sides of length h parallel to the axes of R . When we pass from a network S_N to S_{N+1} each of the squares of S_N splits into four congruent squares.

Let $G \subset R$ be an arbitrary bounded nonempty set. Let the symbol $\omega_N(G) = \omega_N$ denote the figure consisting of all the squares Δ_h of the network S_N which are entirely contained in G and let $\tilde{\omega}_N(G) = \tilde{\omega}_N$ be the figure consisting of those squares Δ_h of S_N each of which contains at least one