S. M. NIKOLSKY

A Course of Mathematical Analysis

Volume

2

MIR PUBLISHERS MOSCOW

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Member, USSR Academy of Sciences

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Volume 2

Translated from the Russian
by
V. M. VOLOSOV, D. Sc.

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TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

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На английском языке

Contents

The major part of this two-volume textbook stems from the course in mathematical analysis given by the author for many years at the Moscow Physico-technical Institute.

The first volume consisting of eleven chapters includes an introduction (Chapter 1) which treats of fundamental notions of mathematical analysis using an intuitive concept of a limit. With the aid of visual interpretation and some considerations of a physical character it establishes the relationship between the derivative and the integral and gives some elements of differentiation and integration techniques necessary to those readers who are simultaneously studying physics.

The notion of a real number is interpreted in the first volume (Chapter 2) on the basis of its representation as an infinite decimal.

Chapters 3–11 contain the following topics: Limit of Sequence, Limit of Function, Functions of One Variable, Functions of Several Variables, Indefinite Integral, Definite Integral, Some Applications of Integrals, Series.

CHAPTER 12

Multiple Integrals

§ 12.1. Introduction

Let us consider a continuous surface, lying in the three-dimensional space with rectangular coordinates (x, y, z), which is determined by an equation

$$z = f(Q) = f(x, y) \qquad (Q = (x, y) \in \Omega)$$

where Ω is a bounded (two-dimensional) set possessing area (two-dimensional measure*). For instance, Ω can be a circle, a rectangle, an ellipse, etc. We shall suppose that the function $f(x, y)$ is positive. Let us state the following problem: it is required to find the volume of the solid bounded above by the given surface and below by the plane $z = 0$, its lateral boundary being the cylindrical surface with generators parallel to the z-axis and passing through the boundary curve γ of the set Ω.

To determine the sought-for volume we resort to the following natural procedure.

The set Ω is divided into a finite number N of parts (subdomains)

$$\Omega_1, \ldots, \Omega_N \tag{1}$$

any two of which either do not intersect or intersect only along some parts of their boundaries. Let these subdomains be such that they possess areas (two-dimensional measures) which we shall denote as $m\Omega_1, \ldots, m\Omega_N$ respectively.

Let us introduce the notion of the diameter of a set: if A is a set in the plane its *diameter* $d(A)$ is defined as

$$d(A) = \sup_{P', P'' \in A} |P' - P''|$$

where the supremum is taken over all the pairs of points P', P'' belonging to A.

Now we choose an arbitrary point $Q_j = (\xi_j, \eta_j)$ $(j = 1, \ldots, N)$ in each part Ω_j and form the sum

$$V_N = \sum_{j=1}^{N} f(Q_j) m\Omega_j \tag{2}$$

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* See § 12.2.

which can be regarded as an approximation to the sought-for volume V. We can naturally suppose that the smaller the diameters $d(\Omega_j)$ of the subdomains Ω_j are, the higher is the accuracy of the approximation $V \approx V_N$. Therefore the *volume* V of the solid in question can be defined as the limit

$$V = \lim_{\max d(\Omega_j)\to 0} \sum_{j=1}^{N} f(Q_j) m\Omega_j \tag{3}$$

to which sum (2) tends when the maximum diameter of the subdomains of partitions (1) are made to tend to zero provided that this limit exists and is independent of the way in which the sequence of partitions (1) is chosen.

Now we can abstract from the problem of finding the volume of a solid and regard expression (3) as the result of an operation performed on the given function f defined in Ω. It is called the *Riemann double integral of the function* f *over the domain* Ω and is denoted

$$V = \lim_{\max d(\Omega_j)\to 0} \sum_{j=1}^{N} f(Q_j) m\Omega_j = \iint_{\Omega} f(x, y)\, dx\, dy = \int_{\Omega} f(Q)\, dQ = \int_{\Omega} f\, d\Omega$$

Let us consider a problem leading to the notion of the triple integral. Suppose that there is a physical body occupying a domain (set) Ω in the three-dimensional space with rectangular coordinates (x, y, z) and that the mass of the body is distributed (nonuniformly, in the general case) over Ω with volume density $\mu\,(x, y, z) = \mu(Q)$ $(Q = (x, y, z) \in \Omega)$. It is required to determine the total mass of the body Ω.

To solve this problem it is natural to partition Ω into N parts $\Omega_1, \ldots, \Omega_N$ whose volumes (three-dimensional measures) are $m\Omega_1, \ldots, m\Omega_N$ (on condition that these volumes exist), to choose an arbitrary point $Q_j = (x_j, y_j, z_j) \in \Omega_j$ $(j = 1, \ldots, N)$ in each of the parts and to define the sought-for mass as the limit

$$M = \lim_{\max d(\Omega_j)\to 0} \sum_{j=1}^{N} \mu(Q_j) m\Omega_j \tag{4}$$

Expression (4) can again be regarded as the result of an operation performed on the function μ defined in the three-dimensional set Ω. It is called the *Riemann triple integral of* f *on* Ω and is denoted as

$$M = \lim_{\max d(\Omega_j)\to 0} \sum \mu(Q_j) m\Omega_j = \int_{\Omega} \mu(Q)\, dQ = \iiint_{\Omega} \mu(x, y, z)\, dx\, dy\, dz$$

The *Riemann n-fold multiple integral* is defined in the same way.

We shall see that the theory of (Riemann) multiple integration which includes existence theorems and theorems on the additive properties of the integral can be presented for the n-dimensional case in exactly the same manner as in the case of dimension 1. However, the theory of multiple integrals involves some specific difficulties which were not encountered in the theorv of one-fold integration.

The matter is that the (Riemann) one-fold integral was defined for an extremely simple set, namely, for a closed interval $[a, b]$ which was partitioned into parts which were also closed intervals. Therefore we had no difficulties in defining the lengths *(one-dimensional measures)* of the intervals. But in the case of a double integral or, generally, n-fold integral, the domain of integration Ω can be split into parts with curvilinear boundaries, which makes it necessary to define the notion of the area or, generally, of the n-dimensional measure of such a part. A similar question would also appear in the case $n = 1$ if we defined the one-fold Riemann integral for a set of a more complex structure than that of a closed interval.

In this connection we must state a strict definition of the notion of measure of a set and investigate the properties of the measure. Therefore we begin this chapter with the theory of the Jordan* measure closely related to the theory of the Riemann integral. This theory forms the basis for the representation of the theory of the Riemann multiple integral. The latter theory provides an important method for evaluation of n-fold multiple integrals by reducing them to the so-called *iterated (repeated) integrals* involving n one-fold integrations with respect to each of the variables; in many important cases this procedure admits of the application of the Newton-Leibniz theorem established for one-fold integrals.

§ 12.2. Jordan Squarable Sets

Let us consider the plane $R = R_2$ with a definitely chosen rectangular coordinates (x, y); this coordinate system will also be denoted by the same letter R.

If some other coordinate system (ξ, η) is taken in the same plane we shall denote the plane (and the new coordinate system) by R'.

A rectangle Δ in the plane R will be regarded as the simplest set. It can be defined analytically by assuming that there is a system of rectangular coordinates R' in which Δ is representable as a set of points (ξ, η) satisfying inequalities of the form

$$a_1 \leqslant \xi \leqslant a_2, \qquad b_1 \leqslant \eta \leqslant b_2 \tag{1}$$

where a_1, a_2, b_1 and b_2 are some numbers such that $a_1 < a_2$ and $b_1 < b_2$. The coordinate system R' possesses the property that the sides of Δ are parallel to its coordinate axes. To stress that the sides of Δ are parallel to the coordinate axes of the system R' we shall write $\Delta = \Delta_{R'}$. The rectangles of the type of Δ are understood here as closed sets (closed rectangles including their boundaries).

Now we define the notion of an *elementary figure* σ*:* a set $\sigma \subset R$ will be called an elementary figure if it is representable as a (set-theoretic) sum of a finite number of rectangles $\Delta \subset R$ any two of which either do not intersect or intersect only along some parts of their boundaries. The area $|\sigma|$ of

* C. Jordan (1838-1922), a French mathematician.

a two-dimensional elementary figure σ is defined as the sum of the areas of the rectangles Δ of which σ is composed.

A given figure σ can be represented as a finite sum (union) of rectangles Δ in infinitely many ways but the area $|\sigma|$ is independent of the representation. This assertion can readily be proved using the means of elementary geometry, and we do not dwell on it here.

An empty set is also regarded as a figure and its measure (area) is understood as being zero.

In inequalities (1) defining a rectangle Δ we assumed that $a_1 < a_2$ and $b_1 < b_2$. Therefore separate points and line segments will not be regarded as rectangles; our representation of the theory of measure will not involve such "degenerate" rectangles.

If an elementary figure σ is representable as a sum of rectangles Δ whose sides are parallel to the axes of the coordinate system R we shall write $\sigma = \sigma_R$.

Enumerated below are some simple properties of elementary figures σ. Their proofs are quite simple and we do not dwell on them here.

(a) *If* $\sigma_1 \subset \sigma_2$ *then* $|\sigma_1| \leqslant |\sigma_2|$.

(b) *The* (set-theoretic) *sum of figures* σ'_R *and* σ''_R *is a figure* σ_R *and there holds the inequality*

$$|\sigma'_R+\sigma''_R| \leqslant |\sigma'_R|+|\sigma''_R|$$

It becomes an equality if σ'_R *and* σ''_R *either do not intersect each other or intersect only along some parts of their boundaries.*

(c) *The difference of two figures* σ'_R *and* σ''_R *is not necessarily a closed set and therefore it may not be an elementary figure. It can only be a figure (possibly empty) if* $\sigma'_R \subset \sigma''_R$ *or if* σ'_R *and* σ''_R *do not intersect. However, the closure* $\overline{\sigma'_R-\sigma''_R}$ *of this difference is always a figure and there holds the inequality*

$$|\overline{\sigma'_R-\sigma''_R}| \geqslant |\sigma'_R|-|\sigma''_R|$$

It turns into an equality if $\sigma''_R \subset \sigma'_R$.

(d) *If a figure* σ_R *is divided into two parts by a line parallel to one of the coordinate axes of the system R these parts are two figures* σ'_R *and* σ''_R.

To these properties we shall add two more; one of them is connected with the notion of a network.

Let us take an arbitrary natural number N and construct two families of straight lines: $x = kh$ and $y = lh$ ($h = 2^{-N}$; $k, l = 0, \pm 1, \pm 2, \ldots$). These families determine the rectangular network S_N dividing R into the squares Δ_h with sides of length h parallel to the axes of R. When we pass from a network S_N to S_{N+1} each of the squares of S_N splits into four congruent squares.

Let $G \subset R$ be an arbitrary bounded nonempty set. Let the symbol $\omega_N(G) = \omega_N$ denote the figure consisting of all the squares Δ_h of the network S_N which are entirely contained in G and let $\tilde{\omega}_N(G) = \tilde{\omega}_N$ be the figure consisting of those squares Δ_h of S_N each of which contains at least one