Robust Control Theory in Hilbert Space

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With 12 Illustrations

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To the memory of Sgan-Aluf David Elkad z"l wounded in action May 18, 1989, died 23 Elul, 5753 (September 9,1993). He served as an example to his soldiers of courage, patriotism, and human decency. May his memory be blessed. H.Y.D.

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Preface

Motivation

The latest texts on linear systems for engineering students have begun incorporating chapters on robust control using the state space approach to H^∞ control for linear finite dimensional time-invariant systems. While the pedagogical and computational advantages of this approach are not to be underestimated, there are, in my opinion, some disadvantages. Among these disadvantages is the narrow viewpoint that arises from the amputation of the finite dimensional time-invariant case from the much more general theory that had been developed using frequency domain methods.

The frequency domain, which occupied center stage for most of the developments of H^∞ control theory, presents a natural context for analysis and controller synthesis for time-invariant linear systems, whether of finite or infinite dimensions. A fundamental role was played in this theory by operator theoretic methods, especially the theory of Toeplitz and skew-Toeplitz operators. The recent lecture notes of Foias, Özbay, and Tannenbaum [3] display the power of this theory by constructing robust controllers for the problem of a flexible beam.

Although controller synthesis depends heavily on the special computational advantages of time-invariant systems and the relationship between H^∞ optimization and classical interpolation methods, it turns out that the analysis is possible without the assumption that the systems are time-invariant.

Why is this of importance? After all, even a complete theory of analysis does not give a student the tools to design a robust stabilizing controller for the most simple system $\dot{y}(t) = u(t)$. There is validity to this point of view, and it is adequately represented by the text mentioned previously. However, research activity in the last few years has shown that the study of time-varying systems using modern mathematical methods has come into its own. This was a scientific necessity. After all, many common physical systems are time-varying. In addition, the use of sample-data methods for the study of linear time-invariant systems requires considering the analysis of time-varying systems. This has been clearly illustrated in the monograph of Chen and Francis [1].

Since this work was begun there has been some progress by various researchers in the development of a controller synthesis theory for time-varying systems. This theory is far from satisfactory at this time, so it is inevitable that there will be significant progress in this direction. The intrinsic difficulty in such a theory is that it must deal with infinite-dimensional infinite-multiplicity models. As my friend and colleague Paul Fuhrmann has said to me many times, only a madman would try to compute in the infinite-multiplicity case. However, the history of mankind has shown that if there is one thing we are never short of, it is madmen.

About This Book

About the same time that Dick Saeks and I were completing our book *System Theory: A Hilbert Space Approach*, I was asked to referee a paper by George Zames with the rather complex title "Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms, and Approximate Inverses." I freely admit that I had no inkling at the time that this paper would be the beginning of a whole new direction in control theory and would, to a great extent, push aside the least-squares optimal control theory presented in our book from an operator theoretic point of view. This paper began what is now called H^∞-control theory and instituted a period of rich interaction between operator theory and optimal control problems.

While the first papers in this subject dealt with the limited format of finite-dimensional linear time-invariant systems, it was quickly seen that, given an operator theoretic formulation, the theory can be naturally stated in an infinite-dimensional setting. The operator theory of Sz-Nagy, Foias commutant lifting, Nevanlinna-Pick interpolation, and Nehari problems of the sixties and seventies suddenly became fundamental issues in optimal control theory. Significant portions of this theory have been presented in an extremely elegant fashion in the monographs of Vidyasagar, Francis, and Foias and Frazho [6], [4], [2].

During 1982 I visited Bruce Francis, and we began discussing formulations of the sensitivity problem in the operator theoretic framework for time-varying systems that had been presented in the Feintuch-Saeks book. It turned out that this framework was very appropriate and that the problems of H^∞ control could be naturally formulated and solved for linear time-varying systems. The key words here were not H^∞ and commutant lifting-Nehari, but nest algebras and Arveson's distance formula. Since the idea was to look at linear systems not pointwise but in a ball, and the appropriate norm was not the Hilbert space norm of the least-squares theory but the operator norm, we called this theory "uniform optimal control."

This book describes the theory of uniform optimal control and robustness of linear time-varying systems using the operator theory framework of nest algebras. Robustness is considered both in the operator norm and in the strongly related gap metric. To make this theory work, a well-developed theory of internal stability was essential, and fortunately this was developed in the last few years as well.

Even though no book on mathematics can be totally self-contained, we tried to at least make the Hilbert space theory as complete as possible. We therefore open with an introductory chapter on the geometry of Hilbert space and the basic notions of operator theory. We have tried to minimize the use of topological notions, and ask the reader's indulgence for the occasional appearance of such terms as cardinality, product topology, nets and weak-compactness. Chapter 2 presents a sequence of more specialized topics chosen by their necessity in the development of the control theory.

Chapters 3 and 4 deal with topics in operator theory as well. These topics, as opposed to the mainly technical ones of Chapter 2, are at the heart of the control problems discussed in the book. Here we present a distance formula for operator matrices, which leads to the formulae of Nehari and Arveson. Another fundamental issue presented here is inner-outer factorization and spectral factorization in nest algebras.

We begin our treatment of linear systems in Chapter 5, where the physical notions of causality are presented in the framework of extended spaces. Chapter 6 studies internal stability of feedback systems. The fundamental idea is to represent a linear system as the range of a 2×1 operator matrix with causal entries. Stabilization is seen to be equivalent to left causal invertibility of such a matrix. This is an appropriate formulation of what is usually called coprime factorization. The classical Youla theorem is presented in this fashion.

Chapter 7 deals with the fundamental problem of uniform optimal control: model matching. We show that a large number of control problems can be presented in this framework as 4-block problems, and we solve such problems.

In Chapters 8 through 10 we present a robust stabilization theory for linear time-varying systems. Chapter 8 deals with coprime factor perturbations and gives formulas for balls of maximal radius that can be stabilized by a single controller and for the structured norm of a system. In Chapter 9 we introduce the gap metric, and Chapters 9 and 10 study robust stabilization in the gap metric. The results of Chapters 8 through 10 are dependent on studying 2-block uniform control problems. Chapter 11 presents the complete solution of the orthogonal embedding problem for passive systems.

I have briefly outlined what is covered in this book. What is not covered is a state-space approach to these problems. This has been described in an elegant manner in the recent book of Halanay and Ionescu [5], and the reader is referred there to the complementary approach to the theory discussed here.

The idea for this book began in a graduate course that I gave on time-varying systems in the spring of 1991. I am fairly sure that my notes for that course would have remained just that if not for the events of January and February 1992. During what is now called "The Gulf War," the State of Israel was attacked by Skud missiles fired by Iraq. Because of the fear of chemical warheads we had to spend long periods of time supervising our families in the vicinity of rooms that were sealed to protect us from the gas. The atmosphere provided by the "mother of all wars" was not conducive to doing new things, but that is where the project of turning my lecture notes into the first draft of this book was begun.

This book is the cumulation of ideas of many people, from whom I learned and with whom I worked on the subjects considered here. A few of those to whom I am particularity grateful are George Zames, Alan Tannenbaum, Malcolm Smith, Tryphon Georgiou, Pramod Khargonekar, Chandler Davis, and Peter Rosenthal. Special thanks to my colleagues Paul Fuhrmann and Alexander Markus. I have left Bruce Francis for special consideration. Without our joint collaborations this project may not have begun.

I have attempted, in the remarks at the end of each chapter, to credit the results of the chapter, whenever possible, to their discoverers. If the credits are not complete, my apologies are rendered to all those whose role was not mentioned.

My wife Sherry and my children, Yonatan, Akiva, Nechama, Noam, and Udi, have had to live with me before, during, and after the writing of this book. This has not always been easy. I express, once again, my gratefulness and love for their infinite patience.

It is appropriate to conclude with the words of Maimonides (Laws of Foundations of the Torah, 2,1):

> Now, what is the way that leads to the love of Him and the reverence for Him? When a person contemplates His great and wonderous acts and creations, obtaining from them a glimpse of his wisdom, which is beyond compare and infinite, he will promptly love and glorify Him, longing exceedingly to know the great Name of God, as David said: 'My whole being thirsts for God, the living God' (Psalms, 42,3). And while pondering over these very subjects he will simultaneously recoil, startled, understanding that he is a lowly obscure creature, as David said: 'As I look up to the heavens thy fingers made what is man that Thou shouldst think of him' (Psalms 8,4-5).

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Contents

1

Basic Hilbert Space Theory

This chapter provides the setting and framework for the rest of the book. It is divided into three parts. The first part concerns itself with the geometry of Hilbert space. The second part gives the basic results in operator theory needed throughout this book, and the third part provides a short introduction to the theory of Banach algebras.

1.1 Geometry of Hilbert Space

Let $\mathcal{H}$ be a complex linear space. A function $(\, , \,) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is called an *inner product* if

1. $(x, x) \geq 0$ for all $x \in \mathcal{H}$ and $(x, x) = 0$ if and only if $x = 0$.
2. $(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 (x_1, y) + \alpha_2 (x_2, y)$ for $\alpha_1, \alpha_2 \in \mathbb{C}$, $x_1, x_2, y \in \mathcal{H}$.
3. $(x, y) = \overline{(y, x)}$.

The pair $\{\mathcal{H}, (\, , \,)\}$ is called an inner product space. From now on we will write the space simply as $\mathcal{H}$ and assume that the function $(\, , \,)$ is known.

We now define a function $\| \; \| : \mathcal{H} \to \mathbb{R}$ by $\|x\| = (x, x)^{\frac{1}{2}}$. Some simple properties of $\|x\|$ follow immediately from its definitions:

1. $\|x\| \geq 0$ for all $x \in \mathcal{H}$ and $\|x\| = 0$ if and only if $x = 0$.
2. $\|\alpha x\| = |\alpha| \, \|x\|$.

Generally, if X is a (real or) complex linear space, a function $\varphi : X \to \mathbb{R}$ is called a seminorm on X if

1′. $\varphi(x) \geq 0$ for all $x \in \mathcal{H}$.

2′. $\varphi(\alpha x) = |\alpha| \, \varphi(x)$.

3′. $\varphi(x + y) \leq \varphi(x) + \varphi(y)$ for all $x, y \in \mathcal{H}$.

If, in addition, $\varphi(x) = 0$ if and only if $x = 0$, then φ is called a norm. It is an important fact that $\| \ \|$ defines a norm on $\mathcal{H}$. To verify this, it must be shown that $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{H}$. We will do this after introducing a few more basic notions.

Two vectors $x, y \in \mathcal{H}$ are *orthogonal* [notation: $x \perp y$] if $(x, y) = 0$. Given a set $\mathcal{M} \subset \mathcal{H}$, x is orthogonal to $\mathcal{M}$ [$x \perp \mathcal{M}$] if $(x, m) = 0$ for all $m \in \mathcal{M}$. A set $\{x_\alpha\}$ is an *orthogonal set* in $\mathcal{H}$ if $(x_\alpha, x_\beta) = 0$, $\alpha \neq \beta$. A vector $x \in \mathcal{H}$ is normalized if $\|x\| = 1$. A set $\{e_\alpha\}$ of vectors in $\mathcal{H}$ is orthonormal if $(e_\alpha, e_\beta) = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$.

For a set $\{e_\alpha\}$ of vectors in $\mathcal{H}$, $\bigvee_\alpha e_\alpha$ is the closure of the subspace they generate.

Theorem 1.1.1 (Pythagorean theorem) *Let $\{x_i\}_{i=1}^n$ be a finite orthogonal set in $\mathcal{H}$. Then*

$$\left\| \sum_{i=1}^n x_i \right\|^2 = \sum_{i=1}^n \|x_i\|^2 .$$

Proof: $\left\|\sum_{i=1}^n x_i\right\|^2 = (\sum_{i=1}^n x_i, \sum_{j=1}^n x_j) = \sum_{i=1}^n \sum_{j=1}^n (x_i, x_j) = \sum_{i=1}^n \|x_i\|^2$. ∎

Theorem 1.1.2 (Schwarz inequality) *For all $x, y \in \mathcal{H}$,*

$$|(x, y)| \leq \|x\| \, \|y\| .$$

Proof: We can assume $y \neq 0$ and let $e = \frac{y}{\|y\|}$. Write x as

$$x = (x, e)e + [x - (x, e)e]$$

and note that $x - (x, e)e \perp e$. Then, by the Pythagorean theorem,

$$\begin{aligned} \|x\|^2 &= \|(x, e)e\|^2 + \|x - (x, e)e\|^2 \\ &\geq \|(x, e)e\|^2 = |(x, e)|^2 . \end{aligned}$$

Thus $|(x, e)| \leq \|x\|$. Now just substitute $e = \frac{y}{\|y\|}$. ∎

Theorem 1.1.3 (triangle inequality) *For all $x, y \in \mathcal{H}$,*

$$\|x + y\| \leq \|x\| + \|y\|.$$

Proof:

$$\begin{aligned} \|x + y\|^2 &= (x + y, x + y) = (x, x) + (x, y) + (y, x) + (y, y) \\ &= \|x\|^2 + \|y\|^2 + 2\mathrm{Re}(x, y) \\ &\leq \|x\|^2 + \|y\|^2 + 2|(x, y)| \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\| \, \|y\| = (\|x\| + \|y\|)^2. \ \blacksquare \end{aligned}$$

It now follows that $\| \ \|$ defines a norm on $\mathcal{H}$. This norm allows us to define a metric topology on $\mathcal{H}$ by means of the metric

$$\rho(x, y) = \|x - y\|.$$

(That this is a metric follows immediately.) A sequence $\{x_n\} \in \mathcal{H}$ is said to converge to $x \in \mathcal{H}$ in the norm (or strongly) on $\mathcal{H}$ if

$$\rho(x_n, x) = \|x_n - x\| \to 0 \text{ as } n \to \infty.$$

We recall that $\mathcal{H}$ is *complete* if every Cauchy sequence in $\mathcal{H}$ converges in $\mathcal{H}$. A complete inner product space is called a *Hilbert space*. A subset $\mathcal{M}$ of $\mathcal{H}$ is called a linear manifold if $x, y \in \mathcal{M}$, $\alpha, \beta \in \mathbf{C}$ implies $\alpha x + \beta y \in \mathcal{M}$. A closed linear manifold is called a *subspace*. Thus a subspace of a Hilbert space is also a Hilbert space.

Theorem 1.1.4 *(,) is continuous simultaneously in both variables.*

Proof: Suppose $\{x_n\}$ converges to x, and $\{y_n\}$ converges to y. Then

$$\begin{aligned} |(x, y) - (x_n, y_n)| &= |(x, y) - (x, y_n) + (x, y_n) - (x_n, y_n)| \\ &\leq |(x, y - y_n)| + |(x - x_n, y_n)| \\ &\leq \|x\| \, \|y_n - y\| + \|x - x_n\| \, \|y_n\| . \end{aligned}$$

Now $\{\|y_n\|\}$ is a convergent sequence converging to $\|y\|$ since $|\, \|y_n\| - \|y\| \,| \leq \|y_n - y\|$. It thus follows that (x_n, y_n) converges to (x, y). ■

Corollary 1.1.5 *If $y \in \mathcal{H}$, then $\{ x \mid (x, y) = 0\}$ is a subspace.*

We have defined the norm in terms of the inner product on $\mathcal{H}$. It is also possible to recover the inner product from the norm.

Theorem 1.1.6 (polarization identity) *For all $x, y \in \mathcal{H}$,*

$$(x, y) = \tfrac{1}{4}\{\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\} .$$

Proof: Compute. ■

Another identity that is quite useful is the *parallelogram law*.

Theorem 1.1.7 *For $x, y \in \mathcal{H}$,*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Proof: Compute. ■

Our next task is to characterize the continuous linear functionals on $\mathcal{H}$. We begin with a result about convex sets in $\mathcal{H}$.

Theorem 1.1.8 *Let $\mathcal{K}$ be a closed convex subset of $\mathcal{H}$, and $x \notin \mathcal{K}$. Then there exists a unique vector $k \in \mathcal{K}$ such that $\|x - k\| \leq \|x - k'\|$ for all k' in $\mathcal{K}$ different from k.*