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Volume 2: Contemporary Analysis

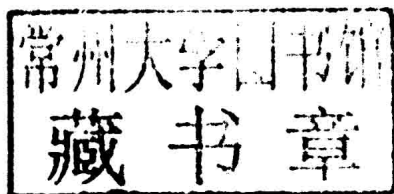
Ali Taheri

Function Spaces and Partial Differential Equations

Volume 2: Contemporary Analysis

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OXFORD
UNIVERSITY PRESS

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Great Clarendon Street, Oxford, OX2 6DP,
United Kingdom

Oxford University Press is a department of the University of Oxford.
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First Edition published in 2015

Impression: 1

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Published in the United States of America by Oxford University Press
198 Madison Avenue, New York, NY 10016, United States of America

British Library Cataloguing in Publication Data
Data available

Library of Congress Control Number: 2014959012

ISBN 978-0-19-873315-7

Printed in Great Britain by
Clays Ltd, St Ives plc

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To Fara and Nick

PREFACE

The origin of this book goes back to the year 2004 when I taught a two semester PhD level course entitled 'Analysis and Partial Differential Equations' in the Mathematical Institute at Oxford. During the intervening years I had the opportunity to teach other courses on related topics to graduate students at Sussex that eventually prompted me to revise and somewhat expand the lecture notes and transform them into the present volumes.

My principal aim in this book is to present a comprehensive treatment of aspects of classical and modern analysis pertaining to *Partial Differential Equations* (PDEs) and their associated *Function Spaces*. I start with a quick review of the basic properties of harmonic functions and Poisson integrals and then move into a detailed study of Hardy spaces. The classical Dirichlet problem is a running theme throughout the book and a variety of methods and techniques for the resolution of the problem ranging from potential theoretic (Perron's method of sub-harmonic functions and Wiener's criterion, Green's functions and Poisson integrals, the method of layered potentials or integral equations) to variational (Dirichlet principle) and Hilbert space methods are presented. Parallel to this is the development of the necessary interpolation and function spaces: Lorentz and Marcinkiewicz spaces, Sobolev spaces (integer as well as fractional order), Hardy spaces, the John-Nirenberg space BMO, Morrey and Campanato spaces, Besov and Triebel-Lizorkin spaces. Harmonic analysis is deeply intertwined with the topics covered and the subjects of summability methods, Tauberian theorems, convolution algebras, Calderon-Zygmund theory of singular integrals and Littlewood-Paley theory that on the one hand connect to various PDE estimates (Calderon-Zygmund inequality, Strichartz estimates, Mihlin-Hormander multipliers) and on the other lead to a unified characterisation of various function spaces are discussed in great depth. The book ends by a discussion of regularity theory for second order elliptic equations in divergence form, first for continuous and then for measurable coefficients. It covers in particular De Giorgi's theorem, Moser iteration, Harnack inequality and local boundedness of solutions. (The case of elliptic systems and related topics is discussed in the exercises at the end.)

The coverage of material is essentially self-contained and there is great attention and emphasis on detail and rigour. The reader is assumed to be conversant in measure theory, functional analysis and spectral theory in a level compatible with most advanced undergraduate or early graduate courses offered in UK universities. Each chapter is accompanied by an extensive collection of examples and hinted exercises of varying level, leading the reader gradually from the basics to the most advanced levels. I hope that by attempting these exercises the reader will not only widen and extend the depth of her mathematical knowledge but also shape and methodically improve her skills, techniques, strength and ability to solve some of the challenging and hard open problems in the field.

One of my main objectives in writing this book is to present and reveal some of the deep and profound inter-connections between different related and seemingly unrelated areas within classical and contemporary analysis. I hope that I have achieved this and the reader working her way through the book will discover these intriguing and fascinating connections for herself. Naturally, for reasons relating to time and space, many sacrifices had to be made and various topics had to be omitted. However at the end of each chapter and often each group of exercises there are further discussions—sometimes lengthy—together with a large collection of references to the literature. I hope that the reader will find these helpful in pursuing her interests beyond the boundaries of the book.

I am indebted to many colleagues and friends with whom over the years I had the opportunity to discuss directly or indirectly various topics relating to the theme of the book. In particular I am grateful to David Applebaum, John Ball, Jonathan Bennett, Jonathan Bevan, Stefano Bianchini, Geoffrey Burton, Miroslav Chlebik, Martin Dindos, Anthony Dooley, David Edmunds, Maria Esteban, Edward Fraenkel, Nicola Fusco, Harold Garth-Dales, Francois Hamel, Qing Han, Emmanuel Hebey, Frédéric Hélein, Tadeusz Iwaniec, Niels Jacob, Bernd Kirchheim, Jan Kristensen, Ari Laptev, Yanyan Li, Fang-Hua Lin, Giuseppe Mingione, Carlos Mora-Corral, Jan Maly, Stefan Müller, Andre Neves, Thomas Schmidt, Jey Sivaloganathan, Alex Sobolev, Endre Suli, Hans Triebel, Nikolaos Tzirakis, Qian Wang, Neshan Wickramasekera, Jim Wright, Po Lam Yung, Arghir Zarnescu and Kewei Zhang.

I am also grateful to my PhD students for their careful reading of various chapters of the book. In particular I thank Abimbola Abolarinwa, Richard Awonusika, Stuart Day, Charles Morris, George Morrison, George Simpson and Mohammad Shahrokhi-Dehkordi.

I would like to take the opportunity to thank my former PhD advisor Sir John Ball FRS for his never failing support, particularly during the early years of my career, and his suggestion of writing this book. I am also grateful to Keith Mansfield and Dan Taber from the Oxford University Press for their highly professional and efficient handling of the manuscript and its publication and to Hemalackshmi Niranjana and Veronica Wastell for their meticulous copy-editing work on the book.

Last but not least I would like to express my deepest gratitude to my wife Faranak and my lovely son Nikan for their understanding, encouragement and support during the time I had to work to finish writing this book and to my mother Shahla and my father Samad for their unwavering support and influence throughout my life.

It is very rewarding for me to see that after all these years, many students and post-docs who took the course in Oxford back in 2004 have gone into research and are now accomplished researchers in the field. I hope that this book will inspire more graduate students and young mathematicians to go into the highly active and exciting fields of Analysis and Partial Differential Equations!

Ali Taheri
Brighton
September 2014

CONTENTS OF VOLUME 1

1 Harmonic Functions and the Mean-Value Property	1
1.1 Spherical Means	1
1.2 Mean-Value Property and Smoothness	4
1.3 Maximum Principles	6
1.4 The Laplace-Beltrami Operator on Spheres	7
1.5 Harnack's Monotone Convergence Theorem	19
1.6 Interior Estimates and Uniform Gradient Bounds	20
1.7 Weyl's Lemma on Weakly Harmonic Functions	23
1.8 Exercises and Further Results	24
2 Poisson Kernels and Green's Representation Formula	33
2.1 The Fundamental Solution N of Δ	34
2.2 Green's Identities and Representation Formulas	36
2.3 The Green's Function $\mathcal{G} = \mathcal{G}(x, y; \Omega)$	41
2.4 The Poisson Kernel $\mathcal{P} = \mathcal{P}(x, y; \Omega)$	44
2.5 Explicit Constructions: Balls	45
2.6 Explicit Constructions: Half-Spaces	52
2.7 The Newtonian Potential $\mathbb{N}[f; \Omega]$	53
2.8 Decay of the Newtonian Potential	59
2.9 Second Order Derivatives and $\Delta \mathbb{N}[f; \Omega]$	61
2.10 Exercises and Further Results	66
3 Abel-Poisson and Fejér Means of Fourier Series	75
3.1 Function Spaces on the Circle	76
3.2 Conjugate Series; Magnitude of Fourier Coefficients	79
3.3 Summability Methods; Tauberian Theorems	82
3.4 Abel-Poisson vs. Fejér Means of Fourier Series	86
3.5 $L^1(\mathbb{T})$ and $\mathcal{M}(\mathbb{T})$ as Convolution Banach Algebras	91
3.6 Approximation to Identity: Strong Convergence in \mathbb{C} and L^p ($p < \infty$)	99
3.7 Approximation to Identity: Weak* Convergence in \mathcal{M} and L^∞	104
3.8 The Riemann-Lebesgue Lemma; An Isomorphism of $L^1(\mathbb{T})$ into $c_0(\mathbb{Z})$	108
3.9 A Primer of Peter-Weyl Theory: Characters and Orthogonality in $L^2(\mathbb{T})$	111
3.10 Exercises and Further Results	114
4 Convergence of Fourier Series: Dini vs. Dirichlet-Jordan	125
4.1 The Wiener Algebra of the Circle $\mathbf{A}(\mathbb{T})$	125
4.2 Pointwise Convergence of Fourier Series	128
4.3 Riemann's Localisation Principle	133
4.4 Dini and Marcinkiewicz Convergence Criteria	133
4.5 Dirichlet-Jordan Convergence Criterion	135
4.6 The Fréchet-Schwartz Space $\mathcal{D}(\mathbb{T})$	137
4.7 The Hilbert-Sobolev Spaces $H^s(\mathbb{T})$ ($-\infty < s < \infty$)	140
4.8 Exercises and Further Results	144

5 Harmonic-Hardy Spaces $h^p(\mathbb{D})$	151
5.1 The Poisson Kernel $\mathcal{P}(x, y; \mathbb{D})$	151
5.2 The Dirichlet Problem in a Jordan Domain	155
5.3 Nodal Sets and the Radó-Kneser-Choquet Theorem	157
5.4 Poisson Integrals in $L^p(\mathbb{T})$ ($1 \leq p \leq \infty$)	162
5.5 Poisson Integrals in $\mathcal{M}(\mathbb{T})$	165
5.6 Non-Tangential Convergence	168
5.7 Characterisation of Harmonic-Hardy Spaces $h^p(\mathbb{D})$	171
5.8 Harmonic Conjugation on $h^p(\mathbb{D})$ ($1 \leq p \leq \infty$)	174
5.9 Hadamard's Three Lines Theorem	177
5.10 Exercises and Further Results	177
6 Interpolation Theorems of Marcinkiewicz and Riesz-Thorin	185
6.1 Interpolation of Integral Operators on $L^p(\mathbb{X}, \mathfrak{A}, \mu)$	185
6.2 Integration via the Distribution Function	192
6.3 Marcinkiewicz Spaces $L_w^p(\mathbb{X}, \mathfrak{A}, \mu)$	195
6.4 Real Interpolation Method of Marcinkiewicz: The Diagonal Case	199
6.5 Complex Interpolation Method of Riesz-Thorin	206
6.6 The Hausdorff-Young and Hardy-Littlewood-Paley Inequalities	212
6.7 Real Interpolation Method of Marcinkiewicz: The General Case	214
6.8 Decreasing Rearrangements; The Maximal Function Operator $\mathcal{M}[f^*]$	219
6.9 The Lorentz Spaces $L^{p,q}(\mathbb{X}, \mathfrak{A}, \mu)$ and Interpolation	225
6.10 Exercises and Further Results	233
7 The Hilbert Transform on $L^p(\mathbb{T})$ and Riesz's Theorem	247
7.1 Fourier Partial Sums and Riesz Projection on $L^p(\mathbb{T})$ ($1 \leq p < \infty$)	247
7.2 Higher Regularity of $u = \mathbb{P}[f]$ Up to the Boundary	251
7.3 The Hilbert Transform on $L^1(\mathbb{T})$; Existence a.e. and Finiteness	253
7.4 The Hilbert Transform as an L^2 -Multiplier Operator	257
7.5 Kolmogoroff's Theorem: The L^1 -weak Estimate on \mathbb{H}	259
7.6 Riesz's Theorem: The L^p -Boundedness of \mathbb{H} ($1 < p < \infty$)	262
7.7 Zygmund's $L \log L$ Theorem and its Converse	265
7.8 Riesz Projection and the L^p -Convergence of Fourier Series ($1 < p < \infty$)	266
7.9 Exercises and Further Results	267
8 Harmonic-Hardy Spaces $h^p(\mathbb{B}^n)$	275
8.1 The Poisson Kernel $\mathcal{P}(x, y; \mathbb{B}^n)$	275
8.2 Poisson Integrals in $L^p(\mathbb{S}^{n-1})$ ($1 \leq p \leq \infty$) and $\mathcal{M}(\mathbb{S}^{n-1})$	279
8.3 Characterisation of Harmonic-Hardy Spaces $h^p(\mathbb{B}^n)$	284
8.4 Herglotz's Theorem on Positive Harmonic Functions	285
8.5 H.A. Schwarz's Reflection Principle; Removable Singularities	286
8.6 Non-Tangential Maximal Function; Stoltz Domains $\Omega_\alpha(y)$ in \mathbb{B}^n	289
8.7 A Spectral Decomposition of $L^2(\mathbb{S}^{n-1})$ via Spherical Harmonics	292
8.8 Orthogonal Projection of $L^2(\mathbb{S}^{n-1})$ onto H_j ; Zonal Harmonics	293
8.9 Exercises and Further Results	299

9 Convolution Semigroups; The Poisson and Heat Kernels on \mathbb{R}^n	309
9.1 Convolutions in $C_0(\mathbb{R}^n)$, $L^p(\mathbb{R}^n)$ and $\mathcal{M}(\mathbb{R}^n)$	309
9.2 $L^1(\mathbb{R}^n)$ and $\mathcal{M}(\mathbb{R}^n)$ as Convolution Banach Algebras	313
9.3 Approximation to Identity: Strong Convergence in C_0 and L^p ($p < \infty$)	316
9.4 Approximation to Identity: <i>Weak*</i> Convergence in \mathcal{M} and L^∞	319
9.5 Young's Convolution Inequality: $L^r(\mathbb{R}^n) \star L^p(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$	320
9.6 Friedrich Mollifiers and Approximation by Smooth Functions	321
9.7 Continuity of Riesz Potentials by way of Young's Inequality	325
9.8 L^p Norm for Vector \mathbf{p} ; The Loomis-Whitney Inequality and Beyond	331
9.9 Exercises and Further Results	334
10 Perron's Method of Sub-Harmonic Functions	351
10.1 Upper Semicontinuous Functions	351
10.2 Sub-Harmonic Functions Revisited	353
10.3 Perron's Existence Theorem	357
10.4 Barriers and the Boundary Regularity of Perron's Solution	359
10.5 Potentials; Capacity and Wiener's Criterion	361
10.6 Harmonic Measure; Generalised Poisson Integrals	374
10.7 The Riemann Mapping Theorem via Green's Functions	376
10.8 Hardy's Theorem on the Convexity of $\log \mathcal{M}_p[f; r]$	378
10.9 Solvability of the Poisson Equation; $C^{2,\alpha}$ Estimates on $\mathbb{N}[f; \Omega]$	380
10.10 Exercises and Further Results	384
11 From Abel-Poisson to Bochner-Riesz Summability	393
11.1 The L^1 Theory of Fourier Transform	393
11.2 Abel-Poisson vs. Gauss-Weierstrass Summability of Integrals	399
11.3 Fourier Inversion Formula on $L^1(\mathbb{R}^n)$	401
11.4 The Schwartz Space $\mathcal{S}(\mathbb{R}^n)$ as a Fréchet Space	404
11.5 Fourier-Plancherel Transform and the L^2 Theory	411
11.6 The Calderon-Zygmund Decomposition Lemma	414
11.7 Summability of Fourier Integrals; Fefferman's Ball Multiplier	416
11.8 Bochner-Riesz Summability	420
11.9 Exercises and Further Results	422
12 Fourier Transform on $\mathcal{S}'(\mathbb{R}^n)$; The Hilbert-Sobolev Spaces $H^s(\mathbb{R}^n)$	437
12.1 $\mathcal{S}'(\mathbb{R}^n)$ as a Dual Space	437
12.2 Fourier Transform on $\mathcal{S}'(\mathbb{R}^n)$	441
12.3 (L^p, L^q) Operators Commuting with Translations	449
12.4 Fractional Integration and $(-\Delta)^{-\alpha/2}$ ($0 < \alpha < n$)	453
12.5 L^p -Estimates: Poisson, Heat and Schrödinger Semigroups	456
12.6 The Wave Kernel W_t ; The Light Cone and Huygens Principle	459
12.7 The Hilbert-Sobolev Spaces $H^s(\mathbb{R}^n)$ ($-\infty < s < \infty$)	462
12.8 Trace Theorems and Restrictions in $H^s(\mathbb{R}^n)$	467
12.9 Extensions and a Theorem of Slobodeckij	468
12.10 Exercises and Further Results	470
<i>Bibliography</i>	481
<i>Index</i>	A1

CONTENTS OF VOLUME 2

13 Maximal Function; Bounding Averages and Pointwise Convergence	501
13.1 A Covering Lemma of Vitali Type	501
13.2 The Hardy-Littlewood Maximal Function	504
13.3 Applications to Differentiability	507
13.4 Approximation to Identity: Pointwise Convergence and Bounds	509
13.5 Local L^1 -Integrability of $\mathbf{M}[f]$ and Stein's $L \log L$ Theorem	516
13.6 L^p -Boundedness of Riesz Potentials via Maximal Function	518
13.7 Young's Convolution Inequality: $L^r_w(\mathbb{R}^n) \star L^p(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$	524
13.8 The Maximal Operator T^* ; Pointwise Convergence of Operator Families $(T_\varepsilon f)$	526
13.9 Exercises and Further Results	530
14 Harmonic-Hardy Spaces $h^p(\mathbb{H})$	549
14.1 The Poisson Kernel $\mathcal{P}(\xi, \zeta; \mathbb{H})$	549
14.2 Poisson Integrals in $L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$) and $\mathcal{M}(\mathbb{R}^n)$	553
14.3 Characterisation of Harmonic-Hardy Spaces $h^p(\mathbb{H})$	555
14.4 Non-Tangential Convergence to Boundary Values	556
14.5 The Hardy-Littlewood Maximal Function on Spheres	559
14.6 Möbius Maps; The Kelvin Transform $K[u]$	564
14.7 Functions Harmonic at Infinity	567
14.8 Positive Harmonic Functions in \mathbb{R}^n_+	574
14.9 Exercises and Further Results	577
15 Sobolev Spaces $W^{k,p}(\Omega)$; A Resolution of the Dirichlet Principle	589
15.1 Calculus of Weak Derivatives	589
15.2 $W^{k,p}$ -Approximation by Smooth Functions	594
15.3 Trace Theorem for $W^{1,p}(\Omega)$; The Zero Trace Space $W^{1,p}_0(\Omega)$	598
15.4 Poincaré Inequality; Equivalent Norms on $W^{k,p}_0$	604
15.5 Gagliardo-Nirenberg-Sobolev Inequality	607
15.6 Embedding Theorems for $W^{k,p}_0$ and $W^{k,p}$	615
15.7 Rellich-Kondrachov Compactness Theorem	620
15.8 The Spectrum of $-\Delta$ and the Perron-Frobenius Theorem	623
15.9 Exercises and Further Results	627
16 Singular Integral Operators and Vector-Valued Inequalities	645
16.1 The Hilbert Transform on $L^p(\mathbb{R})$; Riesz's Theorem by Complex Methods	646
16.2 The Maximal Hilbert Transforms; Riesz's Theorem by Real Methods	651
16.3 Singular Integrals of Calderon-Zygmund Type	657
16.4 The Riesz Transforms \mathbf{R}_j ($1 \leq j \leq n$) on $L^p(\mathbb{R}^n)$ and Beyond	660
16.5 Homogeneous Kernels: L^2 -Boundedness	663
16.6 Homogeneous Kernels: L^p -Theory ($1 \leq p < \infty$)	668

16.7	The Calderon-Zygmund Method of Rotations	670
16.8	Vector-Valued Inequalities; Vector-Valued Singular Integrals	675
16.9	More on the Newtonian Potential $\mathbb{N}[f; \Omega]$	678
16.10	Exercises and Further Results	686
17	Littlewood-Paley Theory, L^p-Multipliers and Function Spaces	701
17.1	Littlewood-Paley Theory on the Line	701
17.2	Littlewood-Paley Theory on the Euclidean n -Space: Part I	706
17.3	Littlewood-Paley Theory on the Euclidean n -Space: Part II	712
17.4	The Hörmander-Mihlin Multiplier Theorem	716
17.5	A Littlewood-Paley Characterisation of $H^s(\mathbb{R}^n)$ and More	718
17.6	Applications to Strichartz Estimates for the Wave Equation	721
17.7	Slobodeckij Spaces $W^{s,p}(\mathbb{R}^n)$ and Bessel Potential Spaces $H_p^s(\mathbb{R}^n)$	727
17.8	Besov Spaces $B_{p,q}^s(\mathbb{R}^n)$ and Triebel-Lizorkin Spaces $F_{p,q}^s(\mathbb{R}^n)$	732
17.9	Embeddings of $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$	734
17.10	Exercises and Further Results	737
18	Morrey and Campanato vs. Hardy and John-Nirenberg Spaces	751
18.1	Morrey Spaces $\mathfrak{M}^{p,\lambda}$	751
18.2	Campanato Spaces $\mathfrak{L}^{p,\lambda}$	753
18.3	Relations Between $\mathfrak{M}^{p,\lambda}$, $\mathfrak{L}^{p,\lambda}$ and $C^{0,\mu}$	754
18.4	The John-Nirenberg Space BMO	760
18.5	The Real Hardy Spaces $\mathcal{H}^p(\mathbb{R}^n)$ ($0 < p \leq \infty$)	765
18.6	$\mathcal{H}^1(\mathbb{R}^n)$ and the Div-Curl Lemma	767
18.7	The $L \log L$ Integrability of $\det \nabla u$ on $W^{1,n}$	770
18.8	Gehring's Higher L^p -Integrability Lemma; Reverse Hölder Inequalities	776
18.9	Exercises and Further Results	779
19	Layered Potentials, Jump Relations and Existence Theorems	799
19.1	The Potential $\mathbb{D} = \mathbb{D}[\phi; \partial\Omega]$ of a Double Layer	799
19.2	The Inner and Outer Trace Operators γ^i, γ^o ; The Jump Relations	810
19.3	The Potential $\mathbb{S} = \mathbb{S}[\phi; \partial\Omega]$ of a Single Layer	811
19.4	The Inner and Outer Trace Operators γ_v^i, γ_v^o ; The Jump Relations	819
19.5	Existence Theorems Through the Method of Layered Potentials	821
19.6	Spectral Analysis of T on $L^2(\partial\Omega)$	822
19.7	An Eigen-Space Decomposition of $L^2(\partial\Omega)$	827
19.8	A Resolution of the Dirichlet and Neumann Problems	829
19.9	Exercises and Further Results	831
20	Second Order Equations in Divergence Form: Continuous Coefficients	841
20.1	Caccioppoli Inequality: The Classical Form	842
20.2	Application to Higher Local Integrability of $ \nabla u ^2$	845
20.3	\mathcal{A} -Harmonic Functions and the Decay Rate of their Integral Means	848
20.4	Comparison with \mathcal{A} -Harmonic Functions; Iteration Lemma	851
20.5	$\mathcal{L}^{2,\lambda}$ -Estimates for \mathcal{A} -Harmonic Functions	853
20.6	Continuous Coefficients: Gradient $\mathfrak{M}^{2,\lambda}$ -Estimates	855
20.7	Gradient Hölder Continuity: $C^{1,\mu}$ -Estimates ($0 < \mu < 1$)	857
20.8	Exercises and Further Results	861

21 Second Order Equations in Divergence Form: Measurable Coefficients	867
21.1 Caccioppoli Inequality on Level Sets	867
21.2 Local Boundedness of Weak Solutions; De Giorgi's Approach	871
21.3 Hölder Continuity of Weak Solutions; Oscillations on Balls	874
21.4 Moser Iteration: Local Boundedness of Weak Solutions	883
21.5 Moser Iteration: Hölder Continuity of Weak Solutions	890
21.6 Harnack Inequality and its Consequences	894
21.7 Exercises and Further Results	898
 Appendices	
A Partition of Unity	907
B Total Boundedness and Compact Subsets of L^p	909
C Gamma and Beta Functions	913
D Volume of the Unit n-Ball: $\omega_n = \mathbb{B}^n$	916
E Integrals Related to Abel and Gauss Kernels	918
F The Hausdorff Measure \mathcal{H}^s ($0 \leq s < \infty$)	922
G Evaluation of Some Integrals Over \mathbb{S}^{n-1}	927
H Sobolev Spaces $W^{1,p}(a, b)$	929
 <i>Bibliography</i>	939
<i>Index</i>	959

