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Function Spaces and Partial Differential Equations

Volume 2: Contemporary Analysis

Ali Taheri

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To Fara and Nick

PREFACE

The origin of this book goes back to the year 2004 when I taught a two semester PhD level course entitled 'Analysis and Partial Differential Equations' in the Mathematical Institute at Oxford. During the intervening years I had the opportunity to teach other courses on related topics to graduate students at Sussex that eventually prompted me to revise and somewhat expand the lecture notes and transform them into the present volumes.

My principal aim in this book is to present a comprehensive treatment of aspects of classical and modern analysis pertaining to *Partial Differential Equations* (PDEs) and their associated *Function Spaces*. I start with a quick review of the basic properties of harmonic functions and Poisson integrals and then move into a detailed study of Hardy spaces. The classical Dirichlet problem is a running theme throughout the book and a variety of methods and techniques for the resolution of the problem ranging from potential theoretic (Perron's method of sub-harmonic functions and Wiener's criterion, Green's functions and Poisson integrals, the method of layered potentials or integral equations) to variational (Dirichlet principle) and Hilbert space methods are presented. Parallel to this is the development of the necessary interpolation and function spaces: Lorentz and Marcinkiewicz spaces, Sobolev spaces (integer as well as fractional order), Hardy spaces, the John-Nirenberg space BMO, Morrey and Campanato spaces, Besov and Triebel-Lizorkin spaces. Harmonic analysis is deeply intertwined with the topics covered and the subjects of summability methods, Tauberian theorems, convolution algebras, Calderon-Zygmund theory of singular integrals and Littlewood-Paley theory that on the one hand connect to various PDE estimates (Calderon-Zygmund inequality, Strichartz estimates, Mihlin-Hormander multipliers) and on the other lead to a unified characterisation of various function spaces are discussed in great depth. The book ends by a discussion of regularity theory for second order elliptic equations in divergence form, first for continuous and then for measurable coefficients. It covers in particular De Giorgi's theorem, Moser iteration, Harnack inequality and local boundedness of solutions. (The case of elliptic systems and related topics is discussed in the exercises at the end.)

The coverage of material is essentially self-contained and there is great attention and emphasis on detail and rigour. The reader is assumed to be conversant in measure theory, functional analysis and spectral theory in a level compatible with most advanced undergraduate or early graduate courses offered in UK universities. Each chapter is accompanied by an extensive collection of examples and hinted exercises of varying level, leading the reader gradually from the basics to the most advanced levels. I hope that by attempting these exercises the reader will not only widen and extend the depth of her mathematical knowledge but also shape and methodically improve her skills, techniques, strength and ability to solve some of the challenging and hard open problems in the field.

One of my main objectives in writing this book is to present and reveal some of the deep and profound inter-connections between different related and seemingly unrelated areas within classical and contemporary analysis. I hope that I have achieved this and the reader working her way through the book will discover these intriguing and fascinating connections for herself. Naturally, for reasons relating to time and space, many sacrifices had to be made and various topics had to be omitted. However at the end of each chapter and often each group of exercises there are further discussions—sometimes lengthy—together with a large collection of references to the literature. I hope that the reader will find these helpful in pursuing her interests beyond the boundaries of the book.

I am indebted to many colleagues and friends with whom over the years I had the opportunity to discuss directly or indirectly various topics relating to the theme of the book. In particular I am grateful to David Applebaum, John Ball, Jonathan Bennett, Jonathan Bevan, Stefano Bianchini, Geoffrey Burton, Miroslav Chlebik, Martin Dindos, Anthony Dooley, David Edmunds, Maria Esteban, Edward Fraenkel, Nicola Fusco, Harold Garth-Dales, Francois Hamel, Qing Han, Emmanuel Hebey, Frédéric Hélein, Tadeusz Iwaniec, Niels Jacob, Bernd Kirchheim, Jan Kristensen, Ari Laptev, Yanyan Li, Fang-Hua Lin, Giuseppe Mingione, Carlos Mora-Corral, Jan Maly, Stefan Müller, Andre Neves, Thomas Schmidt, Jey Sivaloganathan, Alex Sobolev, Endre Suli, Hans Triebel, Nikolaos Tzirakis, Qian Wang, Neshan Wickramasekera, Jim Wright, Po Lam Yung, Arghir Zarnescu and Kewei Zhang.

I am also grateful to my PhD students for their careful reading of various chapters of the book. In particular I thank Abimbola Abolarinwa, Richard Awonusika, Stuart Day, Charles Morris, George Morrison, George Simpson and Mohammad Shahrokhi-Dehkordi.

I would like to take the opportunity to thank my former PhD advisor Sir John Ball FRS for his never failing support, particularly during the early years of my career, and his suggestion of writing this book. I am also grateful to Keith Mansfield and Dan Taber from the Oxford University Press for their highly professional and efficient handling of the manuscript and its publication and to Hemalackshmi Niranjan and Veronica Wastell for their meticulous copy-editing work on the book.

Last but not least I would like to express my deepest gratitude to my wife Faranak and my lovely son Nikan for their understanding, encouragement and support during the time I had to work to finish writing this book and to my mother Shahla and my father Samad for their unwavering support and influence throughout my life.

It is very rewarding for me to see that after all these years, many students and post-docs who took the course in Oxford back in 2004 have gone into research and are now accomplished researchers in the field. I hope that this book will inspire more graduate students and young mathematicians to go into the highly active and exciting fields of Analysis and Partial Differential Equations!

Ali Taheri
Brighton
September 2014

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