TRANSONIC AERODYNAMICS

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PREFACE

Despite recent advances in aerodynamic research for the flight regime lying far beyond the so-called sonic barrier, there remain many stubborn problems connected with those phenomena that occur in the flight range straddling the speed of sound. This transonic region continues to engage the attention of those individuals charged with the design of high speed vehicles: subsonic jets which fly at the brink of the transonic region, and supersonic aircraft and missiles for which passage through the sonic barrier is a critical aspect of flight.

Interest in transonic phenomena is not confined to the jet transport industry. Because of high speed turbines, sonic nozzles, flow-measuring instruments, and other test apparatus based on the principles of transonic flow, this topic is one of widespread interest in fluid mechanics in general.

A by-product of the substantial research effort to resolve the difficulties which abound in this field is the large number of published papers. Although other summary texts have appeared recently in response to the need to organize and digest the voluminous material published in scattered journal articles, the present authors have taken a distinctly different approach in presenting their treatment of the subject.

The main objective pursued has been the unification of this particular field of fluid dynamics by organizing it into a sequential, logical development, in order to cover all the potentially useful approximations that have been suggested for the governing equation of the flow, and which would reveal the interconnections existing between these several approaches. A great deal of attention has been directed toward clarification of basic concepts; in particular, much effort has been expended on explanation of how the shocks of various sorts are generated in the flow, while generous amounts of space have been devoted to the related question of whether there exist shock-free regular (i.e., mathematically well-behaved) transonic flows about airfoil shapes of the continuously turning (i.e., conventional) variety.

Most emphasis has been placed on the theoretical rather than the engineering aspects of the subject. In every instance, however, where any light would be shed on the questions under study by citation of the pertinent experimental results, such comparisons between theory and experiment have been made. In this way the text has acquired a distinctly practical and concrete tenor despite its mathematical orientation.

Having decided to compile something more than a cursory survey, it became evident to the authors in short order that many topics would have to be dropped or scarcely mentioned, even though they might be of special value for certain technical applications; as explained more fully in Section 12 of Chapter VI, however, if such slighted subjects were to be included it would mean that other penalties would be incurred, so that the compromise usually resorted to has been to present only those topics which would be representative of several other particular but related problems.

In our opinion, therefore, the selection of material which was incorporated into the treatise thus evolved ought to have appeal both to mathematicians and to practicing aerodynamicists. This serving of dual masters was not easily achieved, as more than once was brought home to the authors rather painfully when a particularly exasperating struggle was necessary in order to produce a statement of concept or of principle which would be clear and acceptable to both groups of readers. What could be taken for granted by one audience might not be at all familiar to those with different backgrounds, so that some worrisome and difficult choices as to what to include and what to exclude had to be threshed out. It is our conviction, however, that this conscious need to serve the needs of more than a single group of readers has forced us to produce a clearer and more meaningful exposition.

Although the text was a joint undertaking, the responsibility for producing the sections was individually assigned; since, however, there was constant and ample communication between the authors at all times it is believed that maximum integration of the individual contributions has been realized. The mathematical fundamentals given in Chapter III, together with the Appendix to Chapter III concerned with the properties of the hypergeometric functions, was prepared by F. G. Tricomi, while the other chapters were produced by C. Ferrari.

We want to take this opportunity to express our thanks, above all, to the National Research Council of Italy and to its Mathematical Manuscripts Committee for the honor they have bestowed upon us by originally commissioning this book as part of the series they have sponsored.

Turin, Italy

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CHAPTER I

Fundamental Principles

Introduction

The medium that is to be considered in this study is taken to have the following properties:

(a) It is an *ideal* fluid (in the hydrodynamic sense), which means that its viscosity and thermal conductivity are both zero. Under this hypothesis, there can be no transport of momentum or of heat because of velocity gradients or temperature gradients. This supposition is adhered to even though there may be present some discontinuities of either velocity or temperature along certain lines (in two-dimensional flows) or at specific surfaces such as arise when shocks are created in the supersonic part of the stream. Because of this assumption, the entire subject of boundary-layer influences on the inviscid main flow is being avoided.

It must be acknowledged that such viscous and thermal transport processes often can play a decisive role in influencing the behavior of actual flows. It should be made clear at the beginning of this discussion, however, that the principal object of this text is to describe and analyze those special features of flows which are attributable directly to the transonic nature of the flow phenomena, divorced from any essential dependence upon viscous factors.

(b) The medium is a *continuum*, which implies that the molecular constitution of the fluid exerts no influence on the fluid mechanical processes under consideration.

(c) The medium is *homogeneous*, meaning that the physical state of any element of the fluid is determined completely, without the need of specifying any other parameters, once the pressure, density, and temperature are known.

(d) The fluid is a *perfect* gas (in the thermodynamic sense), which amounts to the same thing as asserting that the three state variables (p, pressure; ρ, specific mass, more commonly called the density; and

T, absolute temperature) satisfy at all times the fundamental equation of state

$$p = \rho \frac{R}{m} T = \mathscr{R} \rho T \qquad (\text{thus} \quad \mathscr{R} = R/m)$$

where R is the universal gas constant and m represents 1 mole of the substance under consideration. A mole is the gram molecular weight of the substance, i.e., it is a mass numerically equal to the molecular weight. The gas constant has the generally accepted value of $R = 8.31432$ joules $(°K)^{-1}$ mole^{-1}.

(e) The flow is *transonic*. The term "transonic" designates flows in which the velocity in one region is subsonic, whereas in the remaining part of the flow field it is supersonic. These two regions of the flow are separated from each other by lines (in the two-dimensional case) or by surfaces where the velocity is equal to the velocity of sound; these interfaces are called sonic lines or sonic surfaces. In some instances, the velocity could exhibit a discontinuity at the boundary between the two regions; such a discontinuity constitutes a shock wave.

It is being assumed in this treatment that flow fields that are completely subsonic or completely supersonic have properties that are adequately described in standard texts and, thus, should need no further elaboration here. On occasion, reference will be made to such wholly subsonic or wholly supersonic flows, but only in order to point out how they differ in distinctive ways from the situations met when dealing with the transonic regime.

Furthermore, only a restricted class out of the totality of all types of transonic flow is to be examined. The kinds of transonic flow with which this study will be concerned are to be limited to those cases for which the maximum value of the local Mach number is only slightly past unity [i.e., the flows must obey the stipulation that M (everywhere) $\leqslant M_{max}$, where M_{max} has a value of about 1.3]. This requirement specifically excludes from consideration any high Mach number flows, that is, any flow for which the Mach number M_∞ taken on by the undisturbed freestream at an infinite distance away from any disturbing obstacle is greater than the specified bounding value of 1.3.

The decision to exclude this latter sort of problem from consideration in this analysis is unavoidable, because the two situations are very dissimilar and require completely different lines of approach. Although there is much current interest in cases where a transonic flow is embedded within a generally hypersonic flow field, for which $M_\infty \gg 1$, such combined flows do not allow the principal features of transonic flows to be highlighted and studied with most clarity. If research is to be undertaken with the object of extending into the hypersonic realm the investigations to be made here concerning purely transonic flows, it will be necessary to rescind hypotheses (c) and (d) and to contend with phenomena that are much more complex and less amenable to mathematical description.

1. Equations of Motion; Energy and Entropy Distributions

The equations and general theorems of gas dynamics which are fundamental to the study of the kinds of flow to be examined in what follows have been derived and discussed at length in a large number of well-known texts. For example, the mechanics of gaseous flows is thoroughly treated in several books [1–3].[1] As a convenience to the reader, however, and to aid in establishing a clear set of definitions and in building up a consistent set of principles upon which subsequent work may be firmly based, it is intended in this chapter to present a resume of such background material. Although a few critical steps in the derivations will be indicated, no concerted attempt will be made to prove the laws or to justify the selection of the nomenclature.

The general equations that have to be satisfied in all those areas of the flow where the pressure and density are continuous are as follows:

(a) The *continuity equation.* This law expresses the principle of conservation of mass for the fluid taking part in the motion under examination. The customary formulation for the law is

$$\partial\rho/\partial t + \operatorname{div}(\rho\mathbf{q}) = 0 \tag{1}$$

where $\mathbf{q}$ represents the vector velocity[2] at any point in the field of flow, and t is the time variable.

(b) The *momentum equation.* If it is assumed that all of the external forces acting on the flow are derivable from a potential $\Omega(P, t)$, then the forces and accelerations are related by means of the equation

$$\begin{aligned}\operatorname{grad}\Omega - \frac{1}{\rho}\operatorname{grad}p = \frac{d\mathbf{q}}{dt} &= \frac{\partial\mathbf{q}}{\partial t} + \frac{d\mathbf{q}}{dP}\mathbf{q} \\ &= \frac{\partial\mathbf{q}}{\partial t} + \frac{1}{2}\operatorname{grad}q^2 + \operatorname{rot}\mathbf{q}\times\mathbf{q}\end{aligned} \tag{2}$$

where P is any arbitrary general point in the flow field.

[1] The numbers enclosed in brackets refer to the references listed at the end of each chapter.

[2] Vector quantities are represented by bold-face type, whereas their scalar magnitudes are indicated by the corresponding light-faced type.

The scalar equations that may be written out by taking the components of the several vector quantities appearing in Eq. (2) along the directions of the individual coordinate axes (where spatial locations have been referred to an arbitrarily selected system of orthogonal curvilinear coordinates) are called the Euler equations for the motion.

(c) The *energy equation.* Inasmuch as the gaseous medium being examined here is considered to be thermodynamically perfect, the energy equation is simply

$$\frac{dh}{dt} = \frac{1}{\rho}\frac{dp}{dt} \tag{3}$$

where h is the enthalpy per unit mass. It is better practice to reveal the explicit nature of this derivative by noting that, when f denotes a scalar quantity connected with a specific particle taking part in the fluid motion, then df/dt represents the material (or substantial) derivative of the quantity f. Specifically,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{q} \cdot \operatorname{grad} f.$$

With a slight rearrangement of terms, Eq. (3) may be combined with Eqs. (1) and (2) to result in the basic governing equations of motion:

$$\frac{d}{dt}\left(h + \frac{1}{2}q^2 - \Omega\right) = \frac{1}{\rho}\frac{\partial p}{\partial t} - \frac{\partial \Omega}{\partial t}; \qquad \frac{dE}{dt} = \frac{-p}{\rho}\operatorname{div}\mathbf{q} \tag{4}$$

where the latter relation is obtained by making use of the expression connecting enthalpy and internal energy, E, as given by

$$E = h - p\rho^{-1}.$$

If S is used to represent the entropy, then this quantity may be defined by the equation

$$T\,dS = dE + pd(1/\rho).$$

From Eqs. (4) and (1), it may be decided immediately that under present restrictions

$$dS/dt = 0. \tag{5}$$

This result may be interpreted to mean that along every path followed by any particle in the flow the entropy is constant, provided merely that the paths traced out by the fluid particles are contained within that part of the flow field throughout which the density and pressure vary only in a continuous manner.

Inasmuch as it must be true, from what has been postulated previously, that

$$\rho^{-1} \operatorname{grad} p = \operatorname{grad} h - T \operatorname{grad} S$$

and if, furthermore, the definition of the total energy, H, is introduced by use of the equation

$$H = h + \tfrac{1}{2}q^2 - \Omega,$$

then it will be found by substitution into Eq. (2) that

$$T \operatorname{grad} S = \operatorname{grad} H + \operatorname{rot} \mathbf{q} \times \mathbf{q} + \frac{\partial \mathbf{q}}{\partial t}. \tag{6}$$

This relation prescribes just how the entropy is distributed throughout the flow field. When the flow is steady (so that there is no variation of the local velocities within any time interval, i.e., so that one may take $\partial\mathbf{q}/\partial t = 0$), then this result reduces to the simpler form

$$\mathbf{q} \times \operatorname{rot} \mathbf{q} = \operatorname{grad} H - T \operatorname{grad} S, \tag{6'}$$

which is known as Crocco's equation [4].

2. The Lagrange–Thompson Theorem

The circulation Γ around any closed path l moving with the fluid is represented by the integral

$$\Gamma = \oint_l \mathbf{q} \cdot (ds\, \boldsymbol{\tau}),$$

where the symbol $\oint_l$ denotes the line integral computed around the closed path l and ds indicates an infinitesimal element of distance taken along l, whereas $\boldsymbol{\tau}$ stands for the unit vector lying along the direction of the tangent to l at the point where ds is located. Thus, the rate of change of the circulation is

$$\frac{d\Gamma}{dt} = \oint_l \frac{d\mathbf{q}}{dt} \cdot (ds\, \boldsymbol{\tau}) + \oint_l \mathbf{q} \cdot \frac{d}{dt} (\boldsymbol{\tau}\, ds).$$

The second of these integrals vanishes, however, because it may be readily recognized that

$$\mathbf{q} \cdot \frac{d(\boldsymbol{\tau}\, ds)}{dt} = \frac{\partial}{\partial s} \left(\frac{1}{2} q^2\right) ds.$$

The first of the integrals on the right-hand side of the equation for the rate of change of the circulation may be called the circulation of the acceleration, taken around the closed path l. By reference back to the momentum equation, Eq. (I.1.2),[3] it is clear that this circulation of the acceleration reduces to just

$$\oint_l \frac{d\mathbf{q}}{dt} \cdot (ds\, \boldsymbol{\tau}) = -\oint_l \frac{\operatorname{grad} p}{\rho} \cdot \boldsymbol{\tau}\, ds.$$

Furthermore, since $(dp/\rho) + T\, dS = dh$ it follows that

$$d\Gamma/dt = \oint_l (T\, dS - dh) = \oint_l T\, dS. \tag{1}$$

If it so happens that S is constant throughout the entire field of flow, then the motion is said to be uniformly isentropic, or, better yet, the specific term "homentropic" may be applied. Under this supposition, it is obvious from Eq. (1) that $d\Gamma/dt = 0$ or, thus, the circulation remains constant as time progresses. This result embodies the Lagrange–Thompson theorem, which states, "If the flow is homentropic, then the circulation taken around any moving closed curve in the flow is constant as time proceeds."

3. Flows Starting from Rest, Steady Motion, the Bernoulli Equation, and Other Fundamentals

A powerful theorem concerning flows starting from rest may be deduced from Eq. (I.1.5) when used in conjunction with the hypothesis that there shall be no surfaces of discontinuity in pressure or density at any time in the flow. If a flow starting from rest is defined as one for which $q = 0$ and T is constant for $t = 0$, it is easily deduced, on the basis of the stipulations now made, that

$$S(P, t) = \text{const.} \tag{1}$$

This conclusion means that the flow is homentropic at every point P in the flow field for all subsequent time.

[3] In order to cite formulas in this text, the notation Eq. ($N.i.n$) has been adopted to indicate that reference is being made to Chapter N, Section i, and equation n of the sequentially numbered set of relations to be found there. If, however, the formula to which attention is called is contained within the same section of the text as that in which the citation is made, then the simple notation Eq. (n) is used.

By appeal to the Lagrange–Thompson theorem under these circumstances where the flow starts from rest, it follows that $\Gamma = \text{const} = 0$ for any moving closed curve, because $\Gamma = 0$ at the start of the motion, when $t = 0$.

According to Stokes' theorem, the circulation may be computed by the formula

$$\Gamma = \iint_\sigma \mathbf{n} \cdot \text{rot } \mathbf{q} \, d\sigma$$

where the double integral is to be extended over the entire arbitrary surface σ, which has the closed contour l as a boundary. The symbol $\mathbf{n}$, as usual, denotes the unit vector taken in the direction of the outward normal to the surface σ at any generic point on it. Since the supposition that the flow has started from rest has now led to the condition that

$$\iint_\sigma \mathbf{n} \cdot \text{rot } \mathbf{q} \, d\sigma = 0$$

for any arbitrary surface σ, the conclusion is inescapable that at all locations

$$\text{rot } \mathbf{q} = 0, \tag{2}$$

or it is permissible to make the summary statement that a continuous flow starting from rest will remain free of vorticity thereafter.

If, furthermore, the flow also is steady after a certain instant of time, so that $\partial \mathbf{q}/\partial t = 0$ thenceforth, implying that Eq. (I.1.6′) holds, it follows that

$$h + \tfrac{1}{2} q^2 - \Omega = \text{const} = H_0 \tag{3}$$

where this result may be interpreted to mean that the flow is isoenergetic throughout.

If, in addition, the body forces may be neglected, or if they are nonexistent under this assumed condition of steady flow, the result just obtained becomes even more simple in form because under this further assumption it is true for all points in the flow that

$$H = h + \tfrac{1}{2} q^2 = \text{const} = H_0 . \tag{3′}$$

It is worth pointing out at this juncture that, if a steady flow happens to have streamlines that begin and end at infinity where the motion is uniformly the same on all streamlines, then it is unnecessary to take into consideration the history of the flow preceding the time when it became steady in order to be able to affirm that it will be homentropic, irrotational, and isoenergetic throughout. In fact, under these conditions it may be seen directly from Eqs. (I.1.4)–(I.1.6) that $S = \text{const} = S_0$, $H = \text{const} = H_0$, and $\operatorname{rot} \mathbf{q} = 0$.

The energy form of the Bernoulli equation is expressed by Eq. (3) or (3′) for a steady flow. The latter equation applies when the external forces vanish or are negligibly small, and this assumption about the external forces will be considered to be in effect from now on. The constant H_0 appearing in this form of the Bernoulli equation is the value of h when $q = 0$; consequently, H_0 may be interpreted as the stagnation enthalpy, or, in other words, it is the enthalpy (per unit mass) that is produced in the unit mass as it is subjected to an adiabatic compression (no heat exchanged with other masses) of such magnitude that the flow is brought to a complete stop.

For a perfect fluid having a constant specific heat (in which case it is said to be calorically perfect), the following relations hold true:

$$h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \quad \text{and} \quad H_0 = \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} \tag{4}$$

where γ is the ratio of the specific heat at constant pressure c_p to the specific heat at constant volume c_v. The subscript zero on p and ρ is used to indicate that these are the pressure and density that would be found to exist at a stagnation point. It follows then that

$$a^2 = (\partial p/\partial \rho)_{S=\text{const}} = \gamma p/\rho = \gamma \mathscr{R} T, \tag{5}$$

where a is the velocity of sound in the fluid where the local temperature is T.

Consequently, when the proper substitutions have been carried out, the familiar Bernoulli equation may be put into the alternate forms

$$\frac{\gamma}{\gamma - 1} \frac{p}{\rho} + \frac{1}{2} q^2 = \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0}; \qquad \frac{1}{\gamma - 1} a^2 + \frac{1}{2} q^2 = \frac{1}{2} q_l^2 \tag{6}$$

where the definition has been introduced that

$$q_l^2 = \frac{2\gamma}{\gamma - 1} \frac{p_0}{\rho_0}. \tag{7}$$

The meaning of the quantity q_l may be best understood by observing what happens when the flow is expanded into a vacuum. According to the first of the relations given in Eqs. (6), if the velocity, q, is allowed to increase to such a large value that $p = 0$ (i.e., an absolute vacuum is reached), then $\frac{1}{2} q^2$ equals the constant on the right hand side; consequently, q becomes q_l. Thus, q_l represents the maximum or limit velocity that the flow can attain, provided that the total energy is fixed beforehand.

By examining the second of the relations given in Eqs. (6), the following basic relation becomes evident. If at any point in the gaseous flow the local velocity happens to become equal to the local velocity of sound at the same point, then this velocity, called the critical velocity and customarily denoted by the symbol a^*, is determined by the formula

$$a^{*2} = \frac{\gamma - 1}{\gamma + 1} q_l^2 = \frac{2\gamma}{\gamma + 1} \frac{p_0}{\rho_0} = \frac{2}{\gamma + 1} a_0^2 \tag{8}$$

where a_0 stands for the velocity of sound at a location where the flow has been brought to rest, viz., where the stagnation temperature is T_0.

Some important relations may be expressed in terms of the critical pressure p^*, the critical density ρ^*, and the critical temperature T^*, where these critical properties are the corresponding pressure, density, and temperature that exist at those locations where the local velocity is equal to the critical velocity a^*. In accordance with these definitions, it is seen that

$$a^{*2} = \gamma p^*/\rho^*,$$

with the immediate consequence that

$$\frac{p^*}{\rho^*} = \frac{2}{\gamma + 1} \frac{p_0}{\rho_0}.$$

Inasmuch as it is now being taken for granted that the flow is homentropic, it follows that the pressures and densities are linked by the relation

$$\frac{p^*}{p_0} = \left(\frac{\rho^*}{\rho_0}\right)^\gamma,$$

from which it may be deduced that

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)}; \quad \frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma + 1}\right)^{1/(\gamma-1)}; \quad \frac{T^*}{T_0} = \frac{2}{\gamma + 1}. \tag{9}$$

The value of γ may be determined on the basis of the molecular model envisioned in the classical kinetic theory of gases. According to the tenets of this theory, it follows that $\gamma = 1 + (2/f)$ where f denotes the number of degrees of freedom of the various molecular constructs