Basic Mathematics for Economics, Business, and Finance

E. K. Ummer

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Basic Mathematics for Economics, Business, and Finance

This book can help overcome the widely observed math-phobia and math-aversion among undergraduate students in economics, business, and finance. The book can also help them understand why they have to learn different mathematical techniques, how these techniques can be applied, and how the techniques will equip the students in their further studies.

The book provides a thorough but lucid exposition of most of the mathematical techniques applied in the fields of economics, business, and finance. The book deals with topics from high school mathematics to relatively advanced areas of integral calculus, also covering subjects such as linear algebra, differential calculus, classical optimization, linear and nonlinear programming, and game theory.

Though the book directly caters to the needs of undergraduate students in economics, business, and finance, graduate students in these subjects will also find the book an invaluable text for supplementary reading. The website of the book – www.emeacollege.ac.in/bmebf – provides supplementary materials and further chapters on difference equations, differential equations, elements of Mathematica®, and graphics in Mathematica®. The website also provides materials on the applications of Mathematica® throughout the book, as well as teacher and student manuals.

E. K. Ummer graduated with a Ph.D. degree from the University of London. He is currently an Assistant Professor at the Department of Economics of E.M.E.A. College of Arts and Science. Prior to joining E.M.E.A. College, he was with the faculty of Economics and Management Studies at International Islamic University, Malaysia. His current research interests are quantitative methods, economic theory, and economics of development and growth.

Preface

This book grew out of the lecture notes I gave to students in undergraduate programs in economics, business, and finance for more than a decade at different institutions in different countries. The book can be adopted either wholly or partially for undergraduate or beginning graduate programs in these subjects. The only prerequisite, I assume, to follow the topics covered in this book is a *bit of patience*.

It will not be an exaggeration if one states that mathematics has become the language of economics. The states of affairs in related subjects such as business and finance are not much different. Most of the beginning undergraduate programs in these subjects mainly apply geometric tools for the exposition of relationships and theories. But, as the courses progress, the inherent limitations of the geometric tools necessitate a shift from them to more general algebraic forms. This shift calls for a training in some of the techniques and tools of mathematics.

I have found from experience that an alarmingly large proportion of the students who enroll in undergraduate programs in economics, business, and finance in particular (and social science in general) possess some degree of "math-phobia" and "math-aversion." These feelings, I believe, have their root in the unpropitious presentation of the subject to them. Although the books that have been written on mathematics for these subjects are all excellent in their own respects, many of them still continue the unpropitious form of presentation. Many of these books follow either a notoriously technical or oversimplified approach making the subject esoteric or humdrum. My aim, through an intermediate approach, is an auspicious presentation of the subject so that the feelings of phobia and aversion can be replaced by passion and appreciation. Therefore, I attempt to present to undergraduates in these subjects through this book why they need to learn all the mathematical techniques expected of them; the importance of these techniques and their interrelationships; and how these techniques are applied in their subjects. I believe that this approach will make them appreciate mathematics and, thereby, help them understand their subjects properly.

Most of the graduate programs in economics, business, and finance apply mathematical techniques and tools that are far beyond the levels of those covered in this book. Similarly, most journals (even those considered to be *applied* in nature) appear with articles that contain high-level mathematics. I do not claim either that this book will be sufficient for graduate mathematical requirements or that it will prepare students to read and understand the said articles. But, I do claim that this book is sufficient for undergraduate mathematical requirements and it can build a strong foundation for graduate studies in these subjects, which will eventually help in the reading and understanding of the journal articles mentioned above.

One important feature of the book is that the complete presentation of different topics is based on intuition. Since I believe that visual aids such as graphs help students learn faster, I have included them throughout the book. Though proofs of theorems and propositions are important and necessary for a proper understanding of mathematics, I believe that it will be inauspicious and counterproductive to impose these proofs on already math-phobic and math-averse students. They can learn these proofs once they understood the basics and if they are interested in them. Therefore, I have deliberately omitted the proofs of most of the theorems and propositions. Another feature is that most subsections in every chapter of the book contain a number of numerical examples. Moreover, most major sections of the book contain application examples and exercises from different branches of economics, business, and finance. Although the examples in the book are drawn primarily from these subjects, the main body of the book can be successfully adopted (through suitable selection of examples) in similar programs in subjects such as political science, psychology, life sciences, etc.

The book is organized into eight chapters. I have attempted to include in the book most of the mathematical techniques and tools that are normally taught in undergraduate programs in economics, business, and finance throughout the world. I believe that a review of some of the necessary mathematics learned in school will help students much and, therefore, the *first* chapter of the book is devoted to this purpose. It covers most of the important topics that students learned in school including basics of sets; number system; exponents; logarithms; equations; inequalities; intervals; absolute values; functions; limits; continuity; sequences; series; and sum and product symbols.

The *second* chapter covers linear algebra. This chapter explores most of the topics in vectors and matrices that are required by undergraduate programs in the subjects mentioned above. Specifically, it discusses the basics of vectors and matrices; vector spaces; vector and matrix operations; determinants; inverse; rank; solutions to systems of linear equations; and some special matrices and determinants. Differential calculus is discussed in the *third* chapter. It explores differentiation and derivatives; differentiability of functions; rules of differentiation; higher-order differentiation; curvature of curves; convex sets; transformation of functions and Maclaurin and Taylor series; partial derivatives; differentials; total derivatives; implicit differentiation; etc.

Since static optimization is crucial in the subjects of our interest, chapters four through seven are set apart for this topic. The *fourth* chapter is on classical optimization, which is primarily concerned with the application of linear algebra and differential calculus covered in the second and third chapters, respectively. It begins with a discussion of optima and extrema of univariate functions and progresses through their differential versions; optima of multivariate functions and their extensions; and optimization with equality constraints and its extensions. The *fifth* chapter is devoted to linear programming. The topics covered in this chapter include graphical approach; the tabular and matrix approaches of the simplex method; the revised simplex method; duality and sensitivity analyses; the dual simplex method; transportation and assignment problems; etc. The nonlinear programming approach to optimization is covered in the *sixth* chapter. Topics such as geometric forms of nonlinear objective functions and constraints; geometric and algebraic solutions to nonlinear programming problems; and concave, quasiconcave, and quadratic programming are dealt with in this chapter. Another important topic of static optimization, namely game theory, is dealt with in the *seventh* chapter. It presents topics including static games of complete and perfect information; dominant and dominated strategies; Nash equilibrium; mixed and maximin strategies; dynamic games of complete and perfect information; extensive form representations; subgame perfect Nash equilibrium; repeated games; etc.

The last, and the *eighth*, chapter of the book is devoted to the presentation of one of the tools of dynamic analysis, namely integral calculus. This chapter introduces the meaning of integration; the relationship between integration and differentiation; indefinite integrals; rules of integration; initial value problems; partial and multiple integrals; definite integrals and the fundamental theorem of integral calculus; areas under and between curves; definite partial and multiple integrals; and improper integrals.

I had planned to include in the book, along with a few supplementary topics in the existing chapters, exclusive chapters on difference equations and differential equations. Two issues compelled me to exclude them from the book. One was, of course, the space constraint. The other was the fact that these excluded topics are not widely covered in most undergraduate programs in the subjects of our interest. However, I have prepared these supplementary topics and the two chapters on difference equations and differential equations, which can be found at the book's website: www.emeacollege.ac.in/bmebf. Interested readers can access and use them for learning purposes.

I had also planned to integrate Mathematica®, one of the world's most versatile software packages, into the book. Mathematica is a highly advanced computational software package. It is beyond doubt that it can facilitate students' learning of mathematics. All the figures in this book are generated in Mathematica. But, again, space constraints forced me to exclude it from the book. These materials include exclusive chapters on elements of Mathematica and graphics in Mathematica. They also include the applications of Mathematica in most of the topics covered in the book and in difference and differential equations. These materials can also be found at the website mentioned above and interested readers can use them for learning purposes.

The website also contains teaching aids such as PowerPoint and overhead projector slides; an instructor's manual; and a student solution manual.

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To a certain extent, this book is an outcome of cooperation. Several people have cooperated in many ways during the past few years by providing support, encouragement, guidance, advice, and love. It is certainly a pleasure to acknowledge my debt to them.

This book grew out of the lecture notes I gave to undergraduate students in mathematics for economics, business, and finance at different institutions in different countries. Knowingly or unknowingly, many of these students have contributed greatly to the contents of the book. I am thankful to them all.

My institution, EMEA College of Arts and Science, and my colleagues have been a constant source of support and inspiration. Special thanks are due to Professor K. Abdul Hameed (College Principal); Professor K. Hamza and my other colleagues at the Department of Economics; T.V. Zacharia, Assistant Professor and Head, Department of Political Science; and V. Abdul Muneer, Assistant Professor and Head, Department of Journalism. I am also thankful to A.M. Riyad, Assistant Professor and Head, Department of Computer Science; and to K. Muhammedali, College Superintendent.

Professor P. Mohammed, Head, Department of Mathematics, has patiently reviewed the entire manuscript of the book. His suggestions and comments have certainly improved its contents. I am indebted to him.

It was a great pleasure to deal with the concerned officials at Routledge, Taylor & Francis Group, all the way from the proposal of the book to its present form. In particular, I mention Ms. Lam Yongling, Associate Editor, Routledge. She has been highly cooperative and considerate, and I am grateful to her and all others concerned at Routledge.

My wife, Ani, and my sons, Kittu and Kuttu, have extended me unstinted encouragement and support right from the inception of the idea of writing the book until now. They missed me a lot while I was working away on the book. Although I cannot repay them what is lost, I love them and dedicate the book to them and to all others in my family.

Contents

1 Review of basics

1.1 Introduction

Economic activities have played an important role in the lives of humans for centuries past. We now know that they have an even greater influence on our modern lives. The economic agents in the old civilizations too possessed some perception, though not as sophisticated as we do today, of some of the economic phenomena that affected their lives. But the difference is that they needed only the rudiments of mathematics to analyze and comprehend these phenomena. It was under these circumstances that some of the earliest writers on economics communicated their misty visions.

However, events such as the Renaissance and the Industrial Revolution resulted in radical transformations in production, consumption, trade, and economic management. These transformations are now bolstered by the advent of information technology. These events and the accompanying transformations have made modern economic life highly complex. This suggests that we can no longer be complacent about the rudimentary mathematics that was sufficient until about the beginning of the twentieth century.

One simple example can illuminate the argument we made above. Assume that a consumer wishes to purchase a good offered for sale. But, we are aware of the fact that the consumer's demand for the good depends, *ceteris paribus*, on the price of the good. We know that this is a highly simplified version of reality. In fact, the consumer's demand for the good is also influenced by factors such as the price of related goods (determined in the markets for the related goods); the consumer's income (determined in the factor market); events taking place in the government sector; and so on. Although we started with the simple proposition that a consumer's demand for a good depends on the price of the good, we ended up with a complex situation involving many markets or sectors of the economy.

It would be difficult to analyze such a complex structure as the one presented above without mathematics. The reason is that mathematics can reduce the complexity to manageable limits. Mathematics can help define the elements of a theory precisely; can help generate new insights; and can help in the applicability of the theory. The following view of Fisher (1925: 119), a celebrated American economist, is a testimony to our above statements (italics added):

> *The economic world is a misty region. The first explorers used unaided vision. Mathematics is the lantern by which what before was dimly visible now looms up in firm, bold outlines. The old phantasmagoria*[1] *disappears. We see better. We also see further.*

The above presented necessity generated by the complexity of the economic world paved the way for the advent of mathematics in economic sciences. Mathematics has, in fact, become the language of modern economics, business, and finance. Students of these subjects require a wide variety of mathematical tools of varying degrees of complexity. Since several of the mathematical tools used in these subjects are far beyond the scope of a basic book such as this, we include here only those necessary tools that are required by students for the successful completion of undergraduate programs, and to prepare them for graduate programs, in these subjects.

In this chapter we review some of the essential topics that we will use later. This review will include the basics of topics such as set theory; the number system; exponents; logarithms; equations; inequalities, intervals, and absolute values; relations and functions; limits and continuity; sequences and series; and summation and product notations.

Section 1.2 discusses the fundamental concepts in set theory. This is followed by the number system and the associated properties in Section 1.3. Exponents and their laws are covered in Section 1.4. Section 1.5 reviews logarithms and their properties. A review of the basics of equations is provided in Section 1.6. Section 1.7 presents inequalities, intervals, and absolute values. A review of the fundamental ideas of relations and functions is given in Section 1.8. Limits and continuity are dealt with in Section 1.9. Sequences and series are covered in Section 1.10. We introduce some of the sum and product notations in Section 1.11.

1.2 Set Theory

1.2.1 Meaning of sets

Sets play a crucial role in almost all branches of mathematics and are being increasingly used in economics, business, and finance. It is sometimes convenient to consider many items together. Such a collective entity is called a *set*. A set is defined as any well-defined list, collection, or class of objects. The objects in a set can be anything: students, numbers, vehicles, countries, trees, or anything else. Examples of sets include:

The people living in the city of New York.
The even numbers between 0 and 10.
The odd numbers between 0 and 10.
The numbers 1, 2, 3, 4, and 5.

1.2.2 Set notations

Sets are usually denoted by uppercase letters such as A, B, C, X, Y, Z, etc. The objects in a set are called the *elements* or *members* of the set. These objects are usually denoted by lowercase letters such as a, b, c, x, y, z, etc. If x is an object in the set A, then x is called an element of the set and is denoted as

$x \in A$, and is read "x belongs to A" or "x is a member of A"

If x is not an object in A, then we may write it as

$x \notin A$, and is read "x does not belong to A" or "x is not a member of A"

We can represent a set by listing its elements and using {} notation. Assume that the set A consists of numbers 2, 4, 6, 8, and 10. Then we may write the set A as

$$A = \{2, 4, 6, 8, 10\}$$

Notice that in the set A above we separated the elements by commas and enclosed them in curly brackets. We call this form of representation of a set the *tabular form*. Sets can also be represented by stating properties that its elements must satisfy. Assume that we want a set B of even numbers. Then we may write it as

$$B = \{x \mid x \text{ is even}\}$$

which we read as "B is the set of numbers x such that x is even." This form of representation of a set is called the *set-builder form*.

1.2.3 Equality of sets and subsets

Two sets A and B are said to be equal if they have the same elements; that is, if every element in A also belongs to B and if every element in B also belongs to A. Let $A = \{9, 8, 7, 6\}$ and $B = \{8, 7, 9, 6\}$. Then $A = B$. Notice that a set does not change if its elements are rearranged. Notice also that the set $\{1, 2, 3, 3, 4\} = \{1, 2, 3, 4\}$.

Let there be two sets A and B. If every element in A is also an element of B, then A is called a *subset* of B. In other words, A is a subset of B if $a \in A$ and $a \in B$, and is denoted as $A \subseteq B$. For example, let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$. Since the elements 1, 2, and 3 appear in both sets and since B contains more elements than A does, then $A \subseteq B$. Notice that if $A = B$, $A \subseteq B$ and $B \subseteq A$. Assume that $A \subseteq B$. Then, we may also write $B \supseteq A$, which we read "B is a *superset* of A."

Another term widely used is the *proper subset*. Let there be two sets A and B. Then A is called a proper subset of B if $A \subseteq B$ and $A \neq B$, and is denoted as $A \subset B$. As an example, if $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$, then $A \subset B$.

1.2.4 Types of sets

There are a number of different types of sets. One of the basic types of sets is the *null set* or *empty set*, which is denoted by the Greek letter Φ (phi).[2] As an example, let A be a set of people who are neither dead nor alive. We can write this set using the set-builder for as $A = \{x|x \textit{ is a person who is neither dead nor alive}\}$. We know that this set is a null or empty set. Notice that Φ is considered to be a subset of all other sets.

Sets can be finite or infinite. A set is said to be a *finite set* if it contains a finite number of different elements. Otherwise the set is called an *infinite set*. The set of months in a year, the set of hours in a day, etc., are examples of finite sets. The set of stars in the sky, the set of real numbers, etc., are the examples of infinite sets.

Two other important sets widely used are *universal set* and *complementary set*. The universal set consists of all the objects that are being considered in a particular situation. It is generally denoted by U. The complementary set is the set of all elements that are not the elements of a particular set (say A) but are of U. The complementary set of, say, B is denoted by B'.

Sometimes two or more sets may not have common elements. Such sets are called *disjoint sets*. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$, then A and B are called disjoint sets. Another important type of set is the *power set*. The power set is defined as the set of all the subsets that can be generated from a given set A. It can be shown that if A has n elements, then the power set will contain 2^n elements and is usually denoted as $2^{n(\mathrm{A})}$. For example, let $A = \{1, 2\}$. Then $2^{n(\mathrm{A})} = \{\{1, 2\}, \{1\}, \{2\}, \phi\}$.

1.2.5 Set operations

There are three basic set operations: *union*, *intersection*, and *difference*. We shall review each of them below. The union of two sets A and B is defined as the set of all elements which belong to A, or to B, or to both A and B. We denote the union of sets A and B by $A \cup B$, which is read "A union B." Let $A = \{1, 2, 3, 4\}$ and $B = \{4, 3, 5, 6\}$. Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$.

The intersection of two sets A and B is defined as the set of elements that are common to A and B, and is denoted by $A \cap B$, which is read "A intersection B." In our last example, $A \cap B = \{3, 4\}$.

The difference of two sets A and B is defined as the set of elements which belong to A but not to B and is noted by $A - B$, which is read "A difference B" or "A minus B." In our last example, $A - B = \{1, 2\}$. Notice that $B - A = \{5, 6\}$.

A useful way of representing sets and their operations is the *Venn diagram*, named after the English logician and mathematician John Venn. In a Venn diagram, the universal set U is represented by a square or a rectangle within which individual sets are shown as circles. The Venn diagram representations of union, intersection, difference, and complement are illustrated by the shaded areas in Figures 1.2.1(A)–(D), respectively.

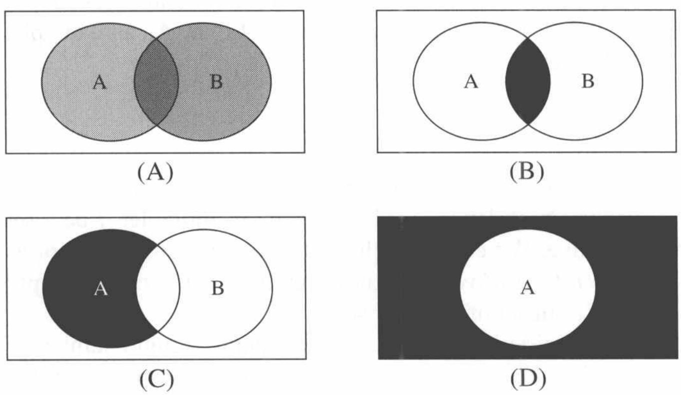
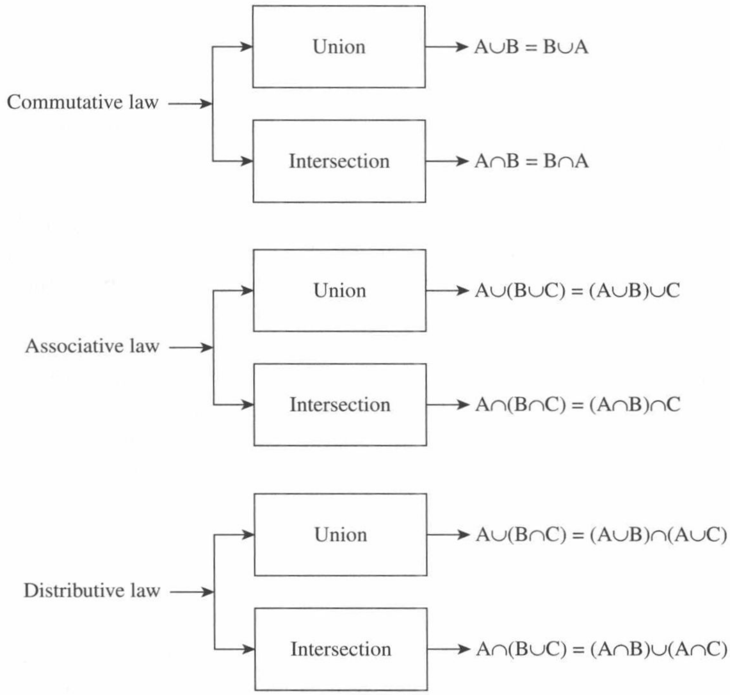


Figure 1.2.1

1.2.6 Laws of set operations

The basic laws of set operations are



1.2.7 Application examples

Example 1. Assume that a company wanted to frame a marketing strategy. The company randomly chose 100 students from the hostels of a university and asked them three questions: (1) Do you have a computer in your room? (2) Do you have a TV in your room? (3) Do you have a computer and a TV in your room? Assume also that 60 of them answered yes to (1), 40 answered yes to (2), and 25 answered yes to (3). (i) How many students have either a computer or a TV in their rooms? (ii) How many students do not have a either a computer or a TV in their rooms? (iii) How many students do have a computer but not a TV in their rooms? (iv) How many students do not have both a computer and a TV in their rooms?

Solution. If we use the Venn diagram, it is easy to solve this problem. But, for this, we need to use specifications such as $U =$ the set of students in the sample (100), $C =$ the set of students who have computers in their rooms (60), $T =$ the set of students who have TV in their rooms (40), and $T \cap C =$ the set of students who have computers and TV in their rooms (25). Now we can use the Venn diagram illustrated in Figure 1.2.2. (i) The number of students who have either a computer or a TV in their rooms is the number of students in the set $T \cup C$. As can be seen from Figure 1.2.2, this number is $35 + 25 + 15 = 75$. (ii) This is equal to the number of students in the set $(T \cup C)'$; that is, $100 - 75 = 25$. (iii) The number of

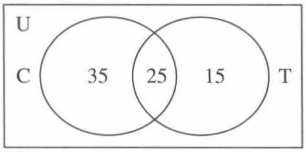


Figure 1.2.2

students who have a computer in their rooms but not a TV is $C - T = 35$. (iv) This is equal to $(T \cap C)' = 75$.

Example 2. Assume that four managers of a company, denoted by the set $\{M_1, M_2, M_3, M_4\}$, wish to select a committee of two people from among themselves. In how many ways can this committee be formed? Or, in other words, how many two-person subsets can be formed from a set of four people?

Solution. Since the elements of the set are M_1, M_2, M_3, and M_4, the subsets with exactly two elements are $\{M_1, M_2\}$, $\{M_1, M_3\}$, $\{M_1, M_4\}$, $\{M_2, M_3\}$, $\{M_2, M_4\}$, and $\{M_3, M_4\}$. This shows that there are six different ways of forming a committee of two managers from among four managers or there are six different subsets of two elements each in a set of four elements.

1.2.8 Exercises

1. Write the following using the tabular form of sets:
   (a) The days in a week. (b) The numbers 1, 2, 3, 4, and 5. (c) The vowels of English alphabet. (d) The South Asian countries India, Pakistan, Sri Lanka, Bangladesh, and Nepal.
2. Continue with exercise 1 above. Write the following using the set-builder form of sets:
   (i) (a); (ii) (b); (iii) (c); (iv) (d).
3. Continue with exercise 1 above. Write the following statements using set notations:
   (i) Sunday is an element of (a); (ii) 6 does not belong to (b); (iii) "b" is not a subset of (c); (iv) India is a subset of (d); (v) Nepal is a proper subset of (d).
4. Let $A = \{a, b, c\}$. Decide whether the following statements are true or false:
   (i) $a \notin A$; (ii) $\{c\} \subseteq A$; (iii) $\{b\} \in A$; (iv) $\{a\} \subset A$; (v) $2^{\mathrm{A}} = 7$.
5. Given the sets $A_1 = \{1, 2, 3\}$, $A_2 = \{5, 1, 3\}$, $A_3 = \{2, 1, 3\}$, and $A_4 = \{3, 1\}$, find:
   (i) $A_1 \cup A_2$; (ii) $A_1 \cap A_2$; (iii) $A_2 \cup A_3$; (iv) $A_2 \cap A_3$; (v) $A_1 \cup A_3$; (vi) $A_1 \cap A_3$; (vii) $A_1 \cup A_2 \cup A_3$; (viii) $A_1 \cap A_2 \cap A_3$.
6. Given $A = \{1, 2, \{3, 4\}, 5\}$, which of the following statements are true and why?
   (i) $\{3, 4\} \subseteq A$; (ii) $\{3, 4\} \in A$; (iii) $\{\{3, 4\}\} \subset A$; (iv) $\{\{3, 4\}, 5\} \not\subset A$.
7. Which of the following statements are valid?
   (i) $A \cup A = A$; (ii) $A \cap A = A$; (iii) $A \cup \phi = A$; (iv) $A \cup U = U$; (v) $A \cap \phi = \phi$; (vi) $A \cap U = U$; (vi) $(A')' = A$.
8. *Application exercise.* A marketing survey of 100 people found that 70 people watch TV news, 40 people listen to radio news, and 30 people both watch TV news and listen to radio news. Find:
   (i) The set of people who watch either TV news or listen to radio news. (ii) The set of people who both watch TV news and listen to radio news. (iii) The set of people who do

not watch either TV news or listen to radio news. (iv) The set of people who do not both watch TV news and listen to radio news. (v) The set of people who watch TV news but do not listen to radio news. (vi) The set of people who listen to radio news but do not watch TV news.

Web supplement: S1.2.9 Mathematica applications

1.3 Number system

Many of the models in the subjects of our interest often use numbers. Moreover, most commercial and financial transactions involve the use of numbers. Therefore, students of economics, business, and finance require knowledge of the fundamental operations involving numbers. Besides, a reasonable understanding of the classification of numbers is also required by these students for further study of mathematics.

1.3.1 Classification of numbers

Numbers are classified into different sets according to certain characteristics. We shall discuss here these sets and their characteristics. Let us begin the classification with *natural numbers*. The natural numbers are also called the *counting numbers*, and we denote them by N. Natural numbers constitute the set of *positive whole numbers*. Therefore, the set of natural numbers is $N = \{1, 2, 3, 4, 5, \ldots\}$. Notice that the natural numbers are closed only under the operations of addition and multiplication. What this means is that when we add or multiply two natural numbers we obtain another natural number. This also means that the difference or quotient of two natural numbers need *not* be a natural number.

Another set of numbers, which is close to the set of natural numbers, is the set of *prime numbers*. The prime numbers are those natural numbers that are only divisible by 1 and by the number itself. We represent the set of prime numbers by P. The set of prime numbers is, therefore, $P = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots\}$.

When we add *negative whole numbers* and zero to the set of natural numbers, we obtain what is called the set of *integers*, denoted by Z. Therefore, the set of integers is written as $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. The integers are also referred to as *whole numbers*. Notice that the integers are closed under the operations of addition, subtraction, and multiplication. This means that the sum, difference, or product of two integers is an integer.

Assume that we divide one integer by another integer (except zero). Then the quotient may or may not be an integer. Such a number is called a *rational number* and is denoted by Q. Therefore, we write the set of rational numbers as $Q = \{x | x = z_1/z_2\}$, where $z_1 \in Z$ and $z_2 \in Z$. It should be noticed that each integer is also a rational number; since, for example, $2/1 = 2$ and so $Z \subset Q$ (i.e. Z is a proper subset of Q). It should also be noticed that rational numbers are closed under all *arithmetic operations*; that is, under addition, subtraction, multiplication, and division. This means that the sum, difference, product, or quotient (except under division by 0) of two rational numbers is a rational number.

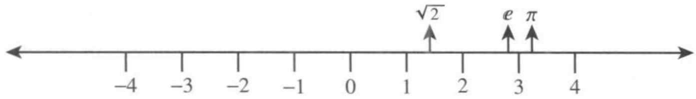


Figure 1.3.1

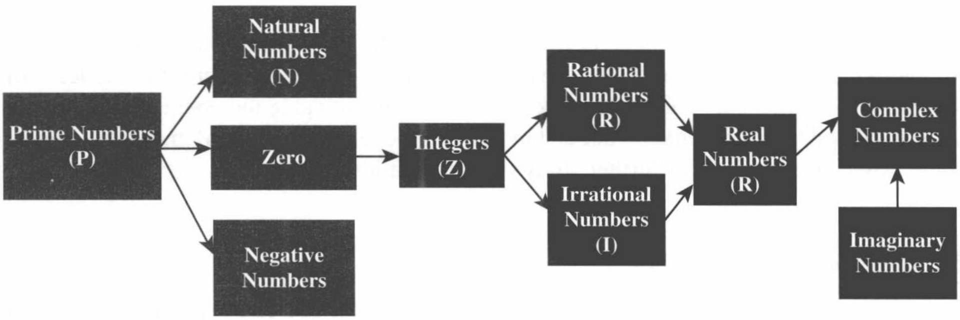


Figure 1.3.2

Can we write every number as the quotient of two numbers? In other words, is every number a rational number? The answer is no. The reason is that some numbers like $\sqrt{2}$, $\sqrt{3}$, e $(= 2.71828\ldots)$, and the value denoted by the Greek letter $\pi (= 3.1415\ldots)$ cannot be written as ratios of integers. The numbers that cannot be written as ratios of integers or the numbers that are not rational numbers are called *irrational numbers*, and we denoted them by I.

One of the most important sets of numbers is the set of *real numbers* denoted by R. The set of all rational and irrational numbers is the set of real numbers. They contain all possible decimal representations. One of the important properties of the real numbers is that they can be represented by points on a straight line. As can be seen from Figure 1.3.1, we choose a point called the *origin* to represent 0 and another point, to the right of 0, to represent 1. Similarly, we choose a point to the left of 0 to represent -1. Then each point will represent a unique real number, and vice versa. We call this line the *real line*. Those numbers to the right of 0 are called the *positive numbers* and those numbers to the left of 0 are called the *negative numbers*. The number 0 is neither positive nor negative.

There is still another set of numbers called *imaginary numbers*. These are the numbers whose squares are negative numbers. $i = \sqrt{-1}$, which implies $i^2 = -1$, is an imaginary number. The last category of numbers is the set of *complex numbers*. Complex numbers have both real and imaginary components and are written in the form $a + bi$, where a and b are real numbers: a is the *real part* and bi is the *imaginary part*. Examples of complex numbers are $2 + 3i$, $10 - 3i$, etc. The above classification of numbers can be represented by a *tree diagram* as illustrated in Figure 1.3.2.