TWELVE LANDMARKS OF Twentieth-Century Analysis

D. Choimet and H. Queffélec

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D. CHOIMET
Lycée du Parc, Lyon

H. QUEFFÉLEC
Université de Lille

Illustrated by
MICHAËL MONERAU

Translated from the French by
DANIÈLE GIBBONS and GREG GIBBONS

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Twelve Landmarks of Twentieth-Century Analysis

The striking theorems showcased in this book are among the most profound results of twentieth-century analysis. The authors' original approach combines rigorous mathematical proofs with commentary on the underlying ideas to provide a rich insight into these landmarks in mathematics. Results ranging from the proof of Littlewood's conjecture to the Banach–Tarski paradox have been selected for their mathematical beauty as well as their educative value and historical role. Placing each theorem in historical perspective, the authors paint a coherent picture of modern analysis and its development, whilst maintaining mathematical rigour with the provision of complete proofs, alternative proofs, worked examples, and more than 150 exercises and solution hints.

This edition extends the original French edition of 2009 with a new chapter on partitions, including the Hardy–Ramanujan theorem, and a significant expansion of the existing chapter on the corona problem.

to our students

Foreword

Analysis... the word is dangerous. Mention it at a dinner party, and depending on your guests, it will bring to mind lab coats and test tubes, or couches and psychoanalysts, or perhaps again those experts that unveil the subtleties of an economical or political crisis. Clarify that you are referring to mathematical analysis and the image will change: former students will then recall memories of derivatives and integrals, and no doubt remind you that it was much easier to calculate the former than the latter... But perhaps one might ask you: Mathematical analysis, no doubt it's all very nice, but what's its point? In fact, what are you analysing?

The book of Denis Choimet and Hervé Queffélec provides brilliant and profound answers to these questions in a most agreeable manner. We follow the evolution of analysis throughout the twentieth century, from the founding fathers Hardy and Littlewood, to the creators of spaces Wiener and Banach and up through contemporaries such as Lennart Carleson. The historical perspective helps us understand the motivation behind the problems, and the naturalness of their solutions. Moreover, analysis is shown clearly for what it is: a discipline situated in the heart of mathematics, indissolubly linked to arithmetic and number theory, to combinatorics, to probability theory, to logic, to geometry... Its objective is hence to serve mathematics and consequently all of the sciences, and thereby each and every one of us.

I have the pleasure of knowing Denis and Hervé. Hence I can assert that their knowledge of analysis can be qualified as encyclopaedic. However, they were not attempting to write an encyclopaedia, and the roots of their work can be found more in Cambridge than in Paris, Warsaw or Moscow. This wise approach allowed them to explore multiple directions right up to the most recent results, while maintaining the profound unity of a very reasonably formatted book, providing constant encouragement to the reader.

Reaching the end, the reader will lay down the book (close at hand, because there are works that demand to be re-read) with the satisfaction of now having a better understanding of analysis. He will also wish to congratulate Denis Choimet and Hervé Queffélec for their collaboration, which illustrates the connectedness of mathematics and of the community of mathematicians. Whether we study them in "classes préparatoires" or in a university, the mathematics stay equally fascinating. Let us not disfigure them by zebra-striping boundaries.

But time for a break from lyricism, to make way for mathematics. Happy reading to all! You are in for a real treat.

Gilles Godefroy, September 2014.

Preface

This book has a history: it was born after the encounter of two professors from different generations, on the occasion of a series of mathematics seminars organised by the younger of the two at the Lycée Clemenceau in Nantes, in the early part of the years 2000 onwards. The prime objective of these seminars was to allow the professors of this establishment to keep a certain mathematical awareness that the sustained rhythm of preparing students for competitive entrance exams did not always facilitate. The seminars took place roughly once a month, and lasted an hour and a half. Over the years, the professors were joined by an increasing number of students from their classes; a vocation for mathematics was born for many of these, possibly in part due to this initiative. Both authors gave half a dozen talks at these seminars, on themes of their choosing, with a strong emphasis (but not exclusively) on classical analysis.

After the nomination of one of us to Lyon, we thought it would be interesting to assemble and write up these talks in more detail, and to find a connection between them. It seemed to us that a good starting point would be the 1911 paper of Littlewood (Chapter 1), which is at the same time the founding point of what we today call Tauberian theorems, and the beginning of the famous collaboration between Hardy and Littlewood that spanned 35 years, until Hardy's death in 1947. This collaboration produced a large number of remarkable discoveries, not the least of which was that of Ramanujan. The magnificent work of Hardy and Ramanujan on the asymptotic behaviour of the partition function is in fact the subject of an entire chapter (Chapter 8).

Some of these discoveries are explained in detail, in addition to the converse of Abel's radial theorem (Chapter 1) – from the functional equation (approximated or not) of the Jacobi θ_0 function and its applications – via Diophantine approximations and continued fractions, to exponential sums and the close study of the "other" function of Riemann (Chapters 7 and 9), and in passing the asymptotic behaviour of the partition function (Chapter 8). Important extensions of this work include Wiener's Tauberian theorem (Chapter 2), the Tauberian theorems of Ikehara and Newman (Chapter 3), which are precursors of Banach algebras and Gelfand theory (Chapter 11), the latter giving rise to the corona problem, brilliantly solved by Carleson in 1962, shortly after his characterisation of interpolation sequences (Chapter 12). Beurling, the supervisor of Carleson (among others), provided a description of the invariant subspaces under the shift operator, a jewel of functional analysis of the twentieth century: this completes the long Chapter 12. Another extension of the study of sums of squares in Chapter 9 is the Littlewood conjecture about the L^1-norm of exponential sums, which was only resolved in 1981 (Chapter 10). A good half of the book thus pays tribute to the English school of analysis and, in passing, to the Swedish and American schools.

A second main theme starts from the work of the Polish school in the 1930s, in particular that of Stefan Banach. The spaces that today are given his name have been the subject of innumerable studies; one of their specific properties, complementation, is described in Chapter 13. This school highly prized the works of the French, in particular those of Baire and Lebesgue. These in turn are well represented in this book, through the study of the generic properties of derivative functions (Chapter 4), generic properties in the domain of probability theory (Chapter 5, which acknowledges the contributions of the Russian school, with Kolmogorov and Khintchine), and finally the paradoxical properties in measure theory (Chapter 6 on the paradox of Banach–Tarski).

All of these works are profound and difficult, but they deserve to be better known and popularised throughout the mathematical community, both from an historical and a scientific point of view. This has been our ambition.

A few words on the style of this book: we did not seek to write a text for highly skilled specialists, thus we were not ashamed to provide many reminders and lots of details and heuristic explanations, and to provide an historical perspective. Nor did we try to write a book for skilled generalists, thus we were not ashamed to provide complete and rigorous proofs, even if very difficult. Therefore, depending on the themes we study, our book spans multiple levels: some portions are at a graduate level, others are at an advanced undergraduate level, the average being somewhere between the two. We thought it useful to extend each chapter with a dozen or so exercises, as a complement to the main text or as an incentive for the reader to continue his reflections. These exercises do not have detailed solutions, but we hope to have provided sufficient references and hints for a reasonably courageous and interested reader to tackle them.

We hope that this book will serve a large audience, even if only now and then: we are thinking of our colleagues, as well as graduate students or *amateurs* of mathematics and beauty (*amateurs* is to be understood as Jean-Pierre Kahane would say).

Our thanks go to our friend Rached Mneimné, whose enthusiasm, openness and efficiency allowed this atypical book to be published, and to Gilles Godefroy, who was kind enough to write a friendly foreword. We sincerely thank our colleagues and former students who accepted reading in depth certain chapters and providing us with precise and constructive feedback: Walter Appel, Frédéric Bayart, Nicolas Bonnotte, Rémi Catellier, Vincent Clapiès, Jean-François Deldon, Quentin Dufour, Jordane Granier, Jérémy Guéré, Denis Jourdan, Xavier Lamy, Stéphane Malek, Thomas Ortiz, Marc Pauly, Michel Staïner, Carl Tipler. We also address warm thanks to the staff at CUP who trusted us and helped us with great kindness and professionalism during the final steps of this translation: Emma Collison, Clare Dennison, Katherine Law, Roger Astley.

Special thanks must be addressed to Bruno Calado and Michaël Monerau. Bruno proofread (in record time) the whole of the manuscript, flushed out an incalculable number of misprints, and proposed a multitude of interesting improvements: without him, the book would not be as polished as we wanted. Michaël not only read in detail several chapters, but also provided 20 or so magnificent diagrams which greatly help in reading and understanding the underlying text, at times extremely technical. It was truly to our benefit that they put their competence at our service.

Last but not least, we could have contemplated translating this new edition by ourselves, producing no doubt masses of "Frenglish". In any case, we totally underestimated the effort involved in translating 500 pages of serious mathematics! As luck would have it, we were introduced to Danièle and Greg Gibbons, both fluent in French, English and . . . mathematics, thus able not only to read our text, but to understand what we were talking about. They did an enormous and excellent job of translating; and for the choice of certain technical terms, it was a true collaboration and a pleasure to discuss with them. Our warmest thanks go to them here.

We welcome with great interest your remarks.

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