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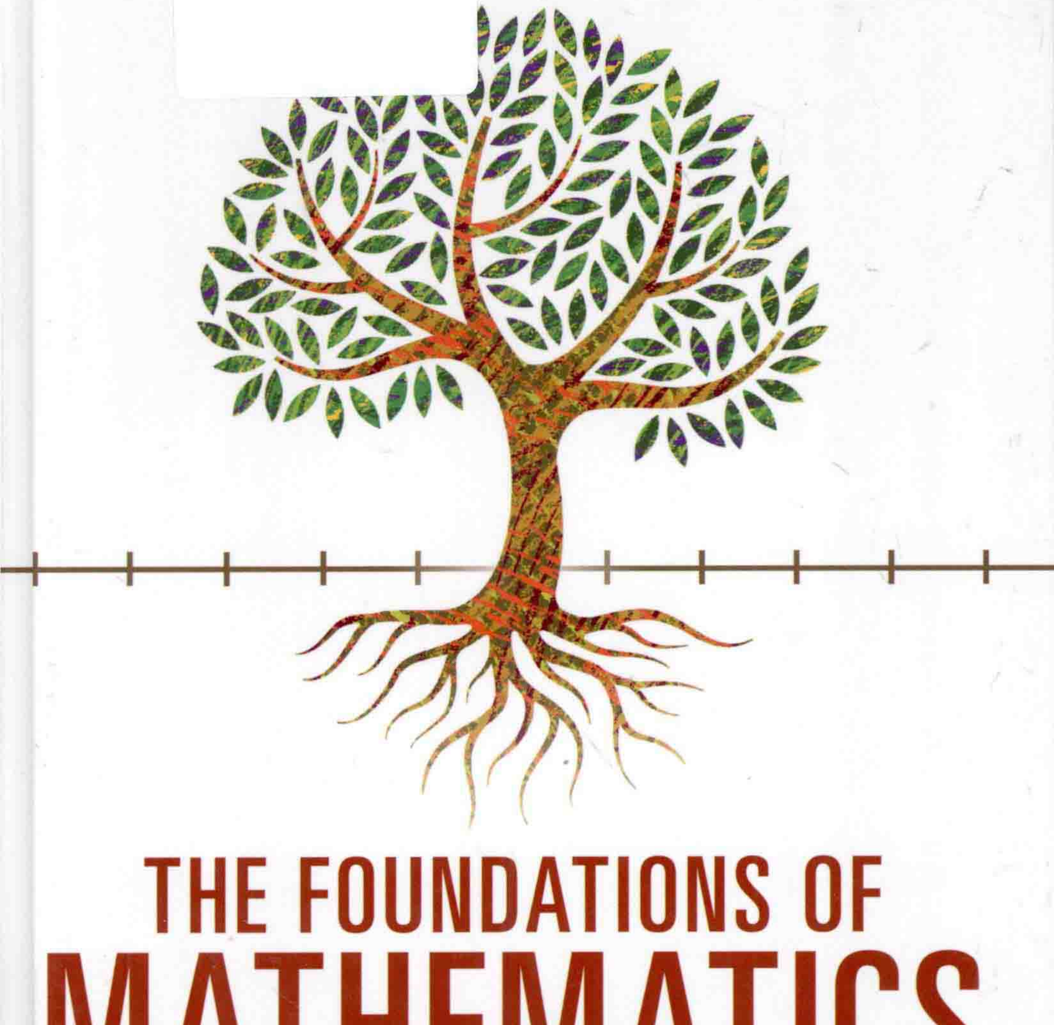

THE FOUNDATIONS OF MATHEMATICS

SECOND EDITION

IAN STEWART
AND DAVID TALL

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Second Edition

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TO
PROFESSOR RICHARD SKEMP

whose theories on the learning of mathematics have been
a constant source of inspiration

PREFACE TO THE SECOND EDITION

The world has moved on since the first edition of this book was written on typewriters in 1976. For a start, the default use of male pronouns is quite rightly frowned upon. Educationally, research has revealed new insights into how individuals learn to think mathematically as they build on their previous experience (see [3]).[1] We have used these insights to add comments that encourage the reader to reflect on their own understanding, thereby making more sense of the subtleties of the formal definitions. We have also added an appendix on *self-explanation* (written by Lara Alcock, Mark Hodds, and Matthew Inglis of the Mathematics Education Centre, Loughborough University) which has been demonstrated to improve long-term performance in making sense of mathematical proof. We thank the authors for their permission to reproduce their advice in this text.

The second edition has much in common with the first, so that teachers familiar with the first edition will find that most of the original content and exercises remain. However, we have taken a significant step forward. The first edition introduced ideas of set theory, logic, and proof and used them to start with three simple axioms for the natural numbers to construct the real numbers as a complete ordered field. We generalised counting to consider infinite sets and introduced infinite cardinal numbers. But we did not generalise the ideas of measuring where units could be subdivided to give an ordered field.

In this edition we redress the balance by introducing a new part IV that retains the chapter on infinite cardinal numbers while adding a new chapter on how the real numbers as a complete ordered field can be extended to a larger ordered field.

This is part of a broader vision of formal mathematics in which certain theorems called *structure theorems* prove that formal structures have natural interpretations that may be interpreted using visual imagination and symbolic manipulation. For instance, we already know that the formal concept of a complete ordered field may be represented visually as points on a number line or symbolically as infinite decimals to perform calculations.

[1] Numbers in square brackets refer to entries in the References and Further Reading sections on page 383.

Structure theorems offer a new vision of formal mathematics in which formal defined concepts may be represented in visual and symbolic ways that appeal to our human imagination. This will allow us to picture new ideas and operate with them symbolically to imagine new possibilities. We may then seek to provide formal proof of these possibilities to extend our theory to combine formal, visual, and symbolic modes of operation.

In Part IV, chapter 12 opens with a survey of the broader vision. Chapter 13 introduces group theory, where the formal idea of a group—a set with an operation that satisfies a particular list of axioms—is developed to prove a structure theorem showing that elements of the group operate by permuting the elements of the underlying set. This structure theorem enables us to interpret the formal definition of a group in a natural way using algebraic symbolism and geometric visualisation.

Following chapter 14 on infinite cardinal numbers from the first edition, chapter 15 uses the completeness axiom for the real numbers to prove a simple structure theorem for any ordered field extension K of the real numbers. This shows that K must contain elements k that satisfy $k > r$ for all real numbers r, which we may call 'infinite elements', and these have inverses $h = 1/k$ that satisfy $0 < h < r$ for all positive real numbers r, which may be called 'infinitesimals'. (There are corresponding notions of negative infinite numbers k satisfying $k < r$ for all negative real numbers r.) The structure theorem also proves that any finite element k in K (meaning $a < k < b$ for real numbers a, b) must be of the form $a+h$ where a is a real number and h is zero or an infinitesimal. This allows us to visualise the elements of the larger field K as points on a number line. The clue lies in using the magnification $m : K \to K$ given by $m(x) = (x - a)/h$ which maps a to 0 and $a + h$ to 1, scaling up infinitesimal detail around a to be able to see it at a normal scale.

This possibility often comes as a surprise to mathematicians who have worked only within the real numbers where there are no infinitesimals. However, in the larger ordered field we can now *see* infinitesimal quantities in a larger ordered field as points on an extended number line by magnifying the picture.

This reveals two entirely different ways of generalising number concepts, one generalising counting, the other generalising the full arithmetic of the real numbers. It offers a new vision in which axiomatic systems may be defined to have consistent structures within their own context yet differing systems may be extended to give larger systems with different properties. Why should we be surprised? The system of whole numbers does not have multiplicative inverses, but the field of real numbers does have multiplicative inverses for all non-zero elements. Each extended system has properties that are relevant to its own particular context. This releases us from the limitations of our real-world experience to use our imagination to develop powerful new theories.

The first edition of the book took students from their familiar experience in school mathematics to the more precise mathematical thinking in pure mathematics at university. This second edition allows a further vision of the wider world of mathematical thinking in which formal definitions and proof lead to amazing new ways of defining, proving, visualising, and symbolising mathematics beyond our previous expectations.

Ian Stewart and David Tall
Coventry 2015

PREFACE TO THE FIRST EDITION

This book is intended for readers in transition from school mathematics to the fully-fledged type of thinking used by professional mathematicians. It should prove useful to first-year students in universities and colleges, and to advanced students in school contemplating further study in pure mathematics. It should also be of interest to a wider class of reader with a grounding in elementary mathematics seeking an insight into the foundational ideas and thought processes of mathematics.

The word 'foundations', as used in this book, has a broader meaning than it does in the building trade. Not only do we base our mathematics on these foundations: they make themselves felt at all levels, as a kind of cement which holds the structure together, and out of which it is fabricated. The foundations of mathematics, in this sense, are often presented to students as an extended exercise in mathematical formalism: formal mathematical logic, formal set theory, axiomatic descriptions of number systems, and technical constructions of them; all carried out in an exotic and elaborate symbolism. Sometimes the ideas are presented 'informally' on the grounds that complete formalism is too difficult for the delicate flowering student. This is usually true, but for an entirely different reason.

A purely formal approach, even with a smattering of informality, is psychologically inappropriate for the beginner, because it fails to take account of the realities of the learning process. By concentrating on the technicalities, at the expense of the manner in which the ideas are conceived, it presents only one side of the coin. The practising mathematician does not think purely in a dry and stereotyped symbolism: on the contrary, his thoughts tend to concentrate on those parts of a problem which his experience tells him are the main sources of difficulty. While he is grappling with them, logical rigour takes a secondary place: it is only after a problem has, to all intents and purposes, been solved intuitively that the underlying ideas are filled out into a formal proof. Naturally there are exceptions to this rule: parts of a problem may be fully formalised before others are understood, even intuitively; and some mathematicians seem to *think* symbolically. Nonetheless, the basic force of the statement remains valid.

The aim of this book is to acquaint the student with the way that a practising mathematician tackles his subject. This involves including the standard 'foundations' material; but our aim is to develop the formal approach as a natural outgrowth of the underlying pattern of ideas. A sixth-form student has a broad grasp of many mathematical principles, and our aim is to make use of this, honing his mathematical intuition into a razor-sharp tool which will cut to the heart of a problem. Our point of view is diametrically opposed to that where (all too often) the student is told 'Forget all you've learned up till now, it's wrong, we'll begin again from scratch, only this time we'll get it right'. Not only is such a statement damaging to a student's confidence: it is also untrue. Further, it is grossly misleading: a student who really did forget all he had learned so far would find himself in a very sorry position.

The psychology of the learning process imposes considerable restraints on the possible approaches to a mathematical concept. Often it is simply not appropriate to *start* with a precise definition, because the content of the definition cannot be appreciated without further explanation, and the provision of suitable examples.

The book is divided into four parts to make clear the mental attitude required at each stage. Part I is at an informal level, to set the scene. The first chapter develops the underlying philosophy of the book by examining the learning process itself. It is not a straight, smooth path; it is of necessity a rough and stony one, with side-turnings and blind alleys. The student who realises this is better prepared to face the difficulties. The second chapter analyses the intuitive concept of a real number as a point on the number line, linking this to the idea of an infinite decimal, and explaining the importance of the completeness property of the real numbers.

Part II develops enough set theory and logic for the task in hand, looking in particular at relations (especially equivalence relations and order relations) and functions. After some basic symbolic logic we discuss what 'proof' consists of, giving a formal definition. Following this we analyse an actual proof to show how the customary mathematical style relegates routine steps to a contextual background—and quite rightly so, inasmuch as the overall flow of the proof becomes far clearer. Both the advantages and the dangers of this practice are explored.

Part III is about the formal structure of number systems and related concepts. We begin by discussing induction proofs, leading to the Peano axioms for natural numbers, and show how set-theoretic techniques allow us to construct from them the integers, rational numbers, and real numbers. In the next chapter we show how to reverse this process, by axiomatising the real numbers as a complete ordered field. We prove that the structures obtained in this way are essentially unique, and link the formal structures to their intuitive counterparts of part I. Then we go on to consider complex numbers, quaternions, and general algebraic and mathematical structures, at which point the whole vista of mathematics lies at our feet. A discussion of infinite cardinals, motivated by the idea of counting, leads towards more advanced work. It also hints that we have not yet completed the task of formalising our ideas.

Part IV briefly considers this final step: the formalisation of set theory. We give one possible set of axioms, and discuss the axiom of choice, the continuum hypothesis, and Gödel's theorems.

Throughout we are more interested in the ideas behind the formal façade than in the internal details of the formal language used. A treatment suitable for a professional mathematician is often not suitable for a student. (A series of tests carried out by one of us with the aid of first-year undergraduates makes this assertion very clear indeed!) So this is not a rigidly logical development from the elements of logic and set theory, building up a rigorous foundation for mathematics (though by the end the student will be in a position to appreciate how this may be achieved). Mathematicians do not think in the orthodox way that a formal text seems to imply. The mathematical mind is inventive and intricate; it jumps to conclusions: it does not always proceed in a sequence of logical steps. Only when everything is understood does the pristine logical structure emerge. To show a student the finished edifice, without the scaffolding required for its construction, is to deprive him of the very facilities which are essential if he is to construct mathematical ideas of his own.

I.S. and D.T.
Warwick
October 1976

CONTENTS

PART I
The Intuitive Background

The first part of the book reflects on the experiences that the reader will have encountered in school mathematics to use it as a basis for a more sophisticated logical approach that precisely captures the structure of mathematical systems.

Chapter 1 considers the learning process itself to encourage the reader to be prepared to think in new ways to make sense of a formal approach. As new concepts are encountered, familiar approaches may no longer be sufficient to deal with them and the pathway may have side-turnings and blind alleys that need to be addressed. It is essential for the reader to reflect on these new situations and to prepare a new overall approach.

Using a 'building' metaphor, we are surveying the territory to see how we can use our experience to build a firm new structure in mathematics that will make it strong enough to support higher levels of development. In a 'plant' metaphor, we are considering the landscape, the quality of the soil, and the climate to consider how we can operate to guarantee that the plants we grow have sound roots and predictable growth.

Chapter 2 focuses on the intuitive visual concept of a real number as a point on a number line and the corresponding symbolic representation as an infinite decimal, leading to the need to formulate a definition for the completeness property of the real numbers. This will lead in the long term to surprising new ways of seeing the number line as part of a wider programme to study the visual and symbolic representations of formal structures that bring together formal, visual, and symbolic mathematics into a coherent framework.