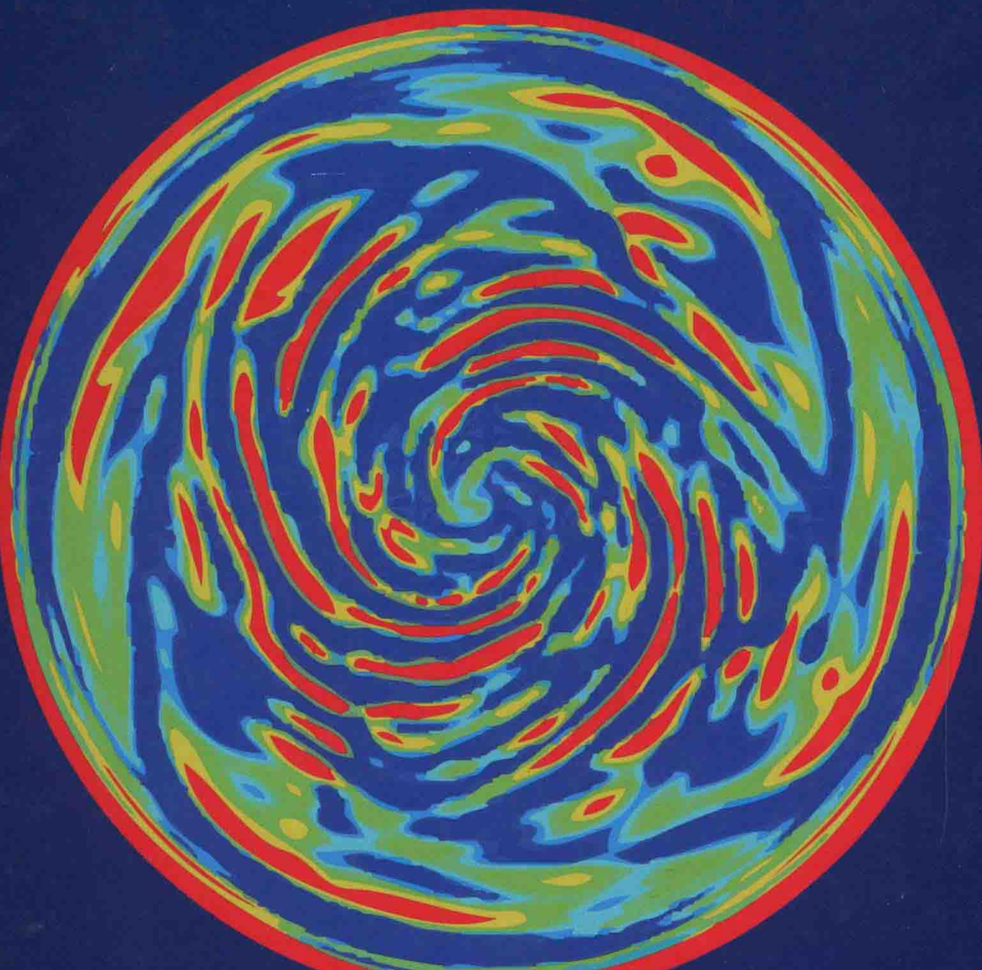
Kemal Hanjalić and Brian Launder

Modelling Turbulence in Engineering and the Environment

Second-Moment Routes to Closure



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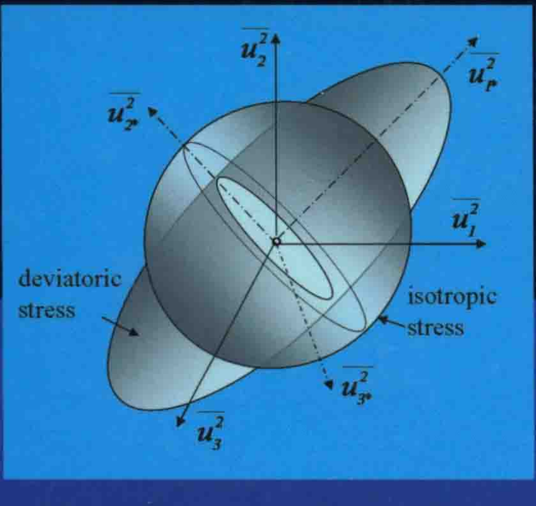
Modelling transport and mixing by turbulence in the complex flows that constantly arise in engineering and in the environment is one of the greatest twenty-first-century challenges for CFD. Yet, too often, because the numerical challenge is also so great, commercial CFD codes adopt simplistic turbulence models that have little validity outside the two-dimensional flows devised in the university laboratory. This highly readable volume introduces the reader to a level of modelling that respects the complexity of the physics of turbulent flows – second-moment closure.

Following introductory chapters providing essential physical background, the book examines in detail the processes to be modelled, from fluctuating pressure interactions to diffusive transport, from turbulent time and length scales to the handling of the semi-viscous region adjacent to walls. It includes extensive examples ranging from fundamental homogeneous flows to three-dimensional industrial and environmental applications. A major chapter examines successive simplification of the models for particular classes of flow, from explicit algebraic second-moment (EASM) closures to linear eddy-viscosity models, including the very simple mixing-length hypothesis. This path enables the reader to see the place of these simpler schemes in the hierarchy of RANS closures and the conditions under which they may be expected to give satisfactory predictions.

This book is ideal for CFD users in industry and academia who seek expert guidance on the modelling options available, and for graduate students in physics, applied mathematics and engineering who wish to enter the world of turbulent flow CFD at the advanced level.



KEMAL HANJALIĆ is Professor Emeritus at the Delft University of Technology in the Netherlands. He has published extensively on the measurement, modelling and simulation of turbulence, including heat transfer, combustion and magneto-fluid dynamics. He is widely recognized as a major contributor to the development of mathematical models of turbulence and served for a decade as chairman of ERCOFTAC's special-interest group on turbulence modelling.

BRIAN LAUNDER is Professor of Mechanical Engineering in the School of Mechanical, Aerospace and Civil Engineering at the University of Manchester. He played a central role in turbulence modelling development, working with his co-author in creating the first widely applied second-moment closure. More recently he has led the application of CFD to three-dimensional turbulent flows, especially in rotating systems, and to the development of the TCL strategy for turbulence modelling.

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Preface

Scientific papers on how to represent in mathematical form the types of fluid motion we call *turbulent flow* have been appearing for over a century while, for the last sixty years or so, a sufficient body of knowledge has been accumulated to tempt a succession of authors to collect, systematize and distil a proportion of that knowledge into textbooks. From the start a bewildering variety of approaches has been advocated: thus, even in the 1970s, the algebraic mixing-length models presented in the book by Cebeci and Smith[1] jostled on the book-shelves with Leslie's[2] manful attempt to make comprehensible to a less specialized readership the direct-interaction approach developed by Kraichnan and colleagues. As the progressive advance in computing power made it possible to apply the emerging strategy of computational fluid dynamics to an ever-widening array of industrially important flows, however, eddy-viscosity models (EVMs) based on the solution of two transport equations for scalar properties of turbulence (essentially, length and time scales of the energy-containing eddies) emerged as the modelling strategy of choice and, correspondingly, have been the principal focus in several textbooks on the modelling of turbulent flows (for example, Launder and Spalding,[3] Wilcox[4] and Piquet[5]).

Today, two-equation EVMs remain the work-horse of industrial CFD and are applied through commercially marketed software to flows of a quite bewildering complexity, though often with uncertain accuracy. However, there has been a major shift among the modelling research community to abandon approaches based on the Reynolds-averaged Navier–Stokes (RANS) equations in favour of

[1] Cebeci, T. and Smith, A. M. O., 1974, *Analysis of Turbulent Boundary Layers*, Ser. Appl. Math. Mech. Vol. XV, Academic Press, New York.
[2] Leslie, D. C., 1973, *Developments in the Theory of Turbulence*, Clarendon Press, Oxford.
[3] Launder, B. E. and Spalding, D. B., 1972, *Mathematical Models of Turbulence*, Academic Press, London.
[4] Wilcox, D. C., 2000, *Turbulence Modeling for CFD*, DCW Industries, Inc., La Cañada, CA.
[5] Piquet, J., 1999, *Turbulent Flows*, Springer, Berlin.

large-eddy simulation (LES) where the numerical solution for any flow adopts a three-dimensional, time-dependent discretization of the Navier–Stokes equations using a model to account simply for the effects of turbulent motions too fine in scale to be resolved with the mesh adopted – that is, a *sub-grid-scale* (or *sgs*) model. While acknowledging that LES offers the prospects of tackling turbulence problems beyond the scope of RANS, a further major driver for this changeover has been the manifold inadequacies of the stress-strain hypothesis adopted by linear eddy-viscosity models. While such a simple linkage between mean strain rate and turbulent stress seemed adequate for a large proportion of two-dimensional, nearly parallel flows, its weaknesses became abundantly clear as attention shifted to recirculating, impinging and three-dimensional shear flows. Although an LES approach will, most probably, also adopt an sgs model of eddy-viscosity type, the consequences are less serious for two reasons. First, the majority of the transport caused by the turbulent motion will be directly resolved by the large eddies and secondly, the finer scale eddies that must still be resolved by the sub-grid-scale model of turbulence will arguably be a good deal closer to isotropy. Thus, adopting an isotropic eddy viscosity as the sgs model may not significantly impair the accuracy of the solution.

However, to overcome many of the weaknesses of linear EVMs used within a RANS framework, it is quite unnecessary to upgrade one's modelling to LES level. Rather than adopting a linear algebraic relation to link stress and strain, one can obtain the turbulent stresses by solving closed forms of the exact Reynolds-stress equations. It is this approach that represents the main focus of the present book, a modelling strategy known formally as *second-moment closure*, a label that also embraces the corresponding modelling of turbulent heat and species fluxes. This closure level, first advocated in the early 1950s,[6] has in principle a far greater capacity than eddy-viscosity models for capturing the diverse influences of complex strain fields, body forces or substantial transport on the evolution of the turbulent stresses. This is because the direct effects of strain field, body forces and convective transport on the turbulent stresses appear directly in the second-moment equations in forms requiring no approximation! It is true that modelling is still needed, both in the second-moment equations and in the scale-determining equation, the latter of which must also be solved to complete closure. But, at second-moment level, one can proceed further by way of analysis while several additional invariant parameters become available to help shape compliance with limiting states of turbulence.

Admittedly, even with a well-constructed code explicitly designed for second-moment closure (as many commercial solvers are not), such schemes require typically twice as much computational resource as corresponding eddy-viscosity

[6] Rotta, J. C., 1951, Statistische Theorie nichthomogener Turbulenz, *Z. Phys.*, **129**, 547 and **131**, 51.

models. But this is a very small price to pay for predicting the flow correctly while the computational costs will still usually be one or two orders of magnitude less than the cost of obtaining an LES of the same flow.

Why, the reader may legitimately ask, if second-moment closure represents such a major advance over eddy-viscosity approaches, has this situation not become evident and widely accepted by potential users? The present authors can offer no certain answer to that question. To those working at that closure level it *is* well known. Indeed, in the more comprehensive current textbooks one will at least find signposts to modern forms of second-moment closures. But perhaps such broad-coverage treatments, while of inestimable value as reference sources, are unable to justify the space for providing a detailed examination of particular modelling forms or for showing a broad coverage of the successes and weaknesses of particular models. Perhaps, we concluded, one needed a textbook that focused principally on second-moment closure, that provided the background in sufficient depth, bringing to light strategies from earlier decades which are still useful and also including the latest models available. Finally, one needed a textbook that discussed in detail a comprehensive range of applications so that potential users could judge the likely utility of the schemes in the flows that interest them. It has been our aim, in the pages that follow, to provide such a coverage.

The writers themselves began working together on second-moment closure in the late 1960s and over the ensuing forty-odd years have repeatedly interacted on research strategy in this field, both in specific collaborative research projects and through the ERCOFTAC[7] special interest group in Turbulence Modelling. Our views on closure modelling, if not identical, are sufficiently closely aligned that, when we learned that each of us was contemplating preparing a textbook on the subject, we quickly decided that we should pool our efforts and produce a joint volume. Throughout, this has been an equal partnership and, as in all our joint papers, our names are sequenced alphabetically.

To a neutral and knowledgeable reader the material presented may well be seen as giving too great an emphasis to the authors' own work. In part this 'bias' arises from wanting to show the performance of particular models for a wide range of test cases that (we have learned from experience) are sensitive to the modelling assumptions. We trust, however, that the cited references make the connection to (and the dependence on) the work of others plainly evident. Indeed, our hope would be that having had their enthusiasm for second-moment closure stimulated or re-awakened by the present text, many readers will be encouraged to plunge into at least some of the other recent textbooks in turbulence modelling and, thereafter, to read the original journal papers that are cited.

[7] European Research Community on Flow Turbulence and Combustion.

In fact, one of the choices made in producing this book is directly aimed at encouraging the reader to progress into the original research literature. In presenting different models, while the main ideas and underlying principles have been included (along with examples of a model's performance), in many cases we have not given a complete mathematical statement still less the boundary conditions or other essential numerical aspects of handling the equations appropriate to different classes of flow.

While, in some respects, the book is more comprehensive in its coverage of second-moment closure than most (perhaps all) alternative volumes on turbulence modelling, there are also omissions about which some brief explanation needs to be given. Although we make early reference to situations where the density fluctuations in the convective transport term need to be acknowledged and modelled, the reader will find that this is not a subject to which we return. The reason is simple: we have ourselves done little work in the area so our position statement could only be arrived at by borrowing conclusions from what others have written. It would, we felt, be better for the interested reader, instead, to digest directly the views of those with greater experience. In fact, two such individuals, Tom Gatski and Jean-Paul Bonnet, have recently collaborated to produce a textbook specifically focused on compressibility in high-speed flow[8] which we commend to the reader. Equally, while both of us have made proposals for obtaining the turbulent thermal time scale by solving an equation for the dissipation rate of temperature fluctuations, we nevertheless nowadays prefer to adopt simpler practices ourselves. Thus, here we leave Nagano's[9] review to summarize the painstaking research and optimization in this area carried out by Nagano and his colleagues. A final important area where we offer no contribution is that of how to embed the concepts of turbulent intermittency within the closure. Long ago Libby[10] proposed a transport equation for intermittency that has been used and developed over the ensuing decades by numerous workers, especially those working in combustion and, more recently, those attempting to predict transition from laminar to turbulent flow. In the latter area the review by Savill[11] gives an indication of the directions being followed to broaden the range of such flows that can be tackled.

Despite the care we have tried to apply in checking the typescript, we know there will inevitably be errors in what is written, whether just typographical slips or interpretational errors on our part. Readers are warmly invited to draw these

8 Gatski, T. B. and Bonnet, J-P., 2009, *Compressibility, Turbulence and High-Speed Flow*, Elsevier, Oxford.

9 Nagano, Y., 2002, Modelling heat transfer in near-wall flows, in *Closure Strategies for Turbulent and Transitional Flows* (Ed. B. E. Launder and N. D. Sandham), 188–247, Cambridge University Press, Cambridge.

10 Libby, P. A., 1975, Prediction of intermittent turbulent flows, *J. Fluid Mech.* **68**, 273–295.

11 Savill, A. M., 2002b, New strategies in modelling by-pass transition, in *Closure Strategies for Turbulent and Transitional Flows* (Ed. B. E. Launder and N. D. Sandham), 493–521, Cambridge University Press, Cambridge.

to our attention (in writing, please) so that in any future re-printing they may be corrected.

In closing, we express our thanks to our host institutions for the infrastructure support they have provided. In the case of KH this also includes La Sapienza University, Rome where, as the holder of an EU-funded Marie Curie Chair, he spent much of the period during the book's preparation. Finally, we are especially conscious that the task of preparing this book would not have been realizable without the contributions of many past and present colleagues. In particular, we offer our thanks and appreciation to Tim Craft, Song Fu, Hector Iacovides, Suad Jakirlić, Saša Kenjereš, Remi Manceau, Kazuhiko Suga and the late Ibrahim Hadžić. We have also benefited greatly over the years from inputs on various aspects of modelling from Peter Bradshaw, Paul Durbin, Tom Gatski, Bill Jones, Nobu Kasagi, Hiroshi Kawamura, Dominique Laurence, Michael Leschziner, John Lumley, Yasu Nagano, Steve Pope, Wolfgang Rodi, Roland Schiestel, Ronald So, Dave Wilcox and Micha Wolfshtein. Finally, we extend a special thank you to the research students and post-doctoral researchers – too numerous to name individually – with whom we have shared the occasional frustrations but, ultimately, the pleasurable satisfactions of turbulence-modelling research.

Kemal Hanjalić, Delft
Brian Launder, Manchester

Nomenclature

A	Lumley's two-component stress ('flatness') parameter, $A \equiv 1 - 9/8(A_2 - A_3)$
A_2	second invariant of stress anisotropy, $A_2 \equiv a_{ij}a_{ji}$
A_3	third invariant of stress anisotropy, $A_3 \equiv a_{ij}a_{jk}a_{ki}$
A_θ	scalar flux correlation function, $A_\theta = (\overline{\theta u_i})^2/(\overline{\theta^2}\,\overline{u_k u_k})$
A^+	coefficient in van Driest's near-wall form of mixing-length hypothesis
a_{ij}	Reynolds-stress anisotropy tensor, $a_{ij} \equiv \overline{u_i u_j}/k - \frac{2}{3}\delta_{ij}$
B_i, $\mathbf{B}$	magnetic flux density
b_{ij}	$b_{ij} = a_{ij}/2$
b_{ki}	Second-order tensor in the model for $\Phi_{\theta i_3}$, Eq (4.86)
b^i_{lj}	third-order tensor in the model for $\Phi_{\theta j_2}$, Eq (4.49)
b^{mi}_{lj}	fourth-order tensor in the model for Φ_{ij_2}, Eq (4.39)
C	species concentration
C_p	pressure coefficient, $2(P_w - P_\infty)/\rho U^2_\infty$
C_κ	constant in Kolmorov's $-\frac{5}{3}$ law for energy variation with wave number, Eq (3.6)
$\mathcal{C}_{ij}$	convection of the Reynolds stress tensor $\overline{u_i u_j}$
$\mathcal{C}_{\theta i}$	convection of the turbulent scalar flux $\overline{\theta u_i}$
$\mathcal{C}_{\theta\theta}$	convection of scalar variance $\overline{\theta^2}$
$\mathcal{C}_\phi$	convection of a turbulence variable ϕ
c_μ	coefficient in eddy-viscosity formula
c_p	specific heat at constant pressure
$c_{\varepsilon 1}, c_{\varepsilon 2}, \ldots$	coefficients of source/sink terms in the modelled ε-equation
$c_1, c_2, \ldots$	coefficients in the models of the pressure-strain term
D	diameter, channel width
D_{ij}	complementary stress production tensor, $D_{ij} \equiv -\left(\overline{u_i u_k}\partial U_k/\partial x_j + \overline{u_j u_k}\partial U_k/\partial x_i\right)$

$\mathcal{D}_{ij}$	total diffusion of the Reynolds stress tensor
$\mathcal{D}^p_{ij}$	turbulent diffusion of the Reynolds stress tensor $\overline{u_i u_j}$ by pressure fluctuations, Eq (2.20)
$\mathcal{D}^t_{ij}$	turbulent diffusion of the Reynolds stress tensor $\overline{u_i u_j}$ by velocity fluctuations, Eq (2.18)
$\mathcal{D}^\nu_{ij}$	molecular diffusion of the Reynolds stress tensor $\overline{u_i u_j}$, Eq (2.18)
$\mathcal{D}_{\theta i}$	total diffusion of scalar flux $\overline{\theta u_i}$, Eq (2.22)
$\mathcal{D}^p_{\theta i}$	turbulent diffusion of scalar flux $\overline{\theta u_i}$ by pressure fluctuations, Eq (2.22, 2.25)
$\mathcal{D}^t_{\theta i}$	turbulent diffusion of scalar flux $\overline{\theta u_i}$ by velocity fluctuations, Eq (2.22, 2.25)
$\mathcal{D}^\alpha_{\theta i}$	thermal molecular diffusion of scalar flux $\overline{\theta u_i}$, Eq (2.22, 2.25)
$\mathcal{D}^\nu_{\theta i}$	viscous diffusion of scalar flux $\overline{\theta u_i}$, Eq (2.22, 2.25)
$\mathcal{D}_{\theta\theta}$	total diffusion of scalar variance $\overline{\theta^2}$, Eq (3.20)
$\mathcal{D}_\phi$	total diffusion of a turbulence variable ϕ
$\mathcal{D}^p_\phi$	turbulent diffusion of variable ϕ by pressure fluctuations
$\mathcal{D}^t_\phi$	turbulent diffusion of variable ϕ by velocity fluctuations
$\mathcal{D}^\nu_\phi$	molecular diffusion of variable ϕ
E	two-component-limit parameter for dissipation tensor, $E = 1 - \frac{9}{8}(E_2 - E_3)$
E	integration constant in log-law, $E \approx 8.4$ for a smooth wall
E_2	second invariant of e_{ij}, $E_2 = e_{ij}e_{ji}$
E_3	third invariant of e_{ij}, $E_3 = e_{ij}e_{jk}e_{ki}$
$E(\kappa)$	contribution by the Fourier-mode wave number κ to the turbulent kinetic energy
e_i	fluctuating electric potential
e_{ij}	stress dissipation-rate anisotropy tensor, $e_{ij} \equiv \varepsilon_{ij}/\varepsilon - \frac{2}{3}\delta_{ij}$
$\mathcal{F}_{ij}$	turbulent stress flux production due to all body forces, Eq (2.19)
$\mathcal{F}_{\theta i}$	turbulent scalar flux production due to all body forces, Eq (2.23)
$\mathcal{F}_\phi$	production of a turbulence variable ϕ by all body forces
f	scalar variable in Durbin's elliptic relaxation EVM
f_i	fluctuating body force
f_w	wall damping function in GL and HJ low-Re RSM
$\mathcal{G}_{ij}$	turbulent stress production due to gravitational force, Eqs (2.19, 4.74)
g	gravitational acceleration constant
g_i, $\mathbf{g}$	gravitational vector
H	height of the step in flow over a backward-facing step

H, H_{12}	boundary-layer shape factor, δ^*/θ (note $\delta_1 \equiv \delta^*$, $\delta_2 \equiv \theta$, $H_{12} \equiv H$)
Ha	Hartmann number
h	half-width of a plane channel
h	enthalpy, $h \equiv \int c_p dT$
II	alternative notation for the second invariant of stress anisotropy, $II \equiv b_{ij}b_{ji}/2 = A_2/8$
III	alternative notation for the third invariant of stress anisotropy $III \equiv b_{ij}b_{jk}b_{ki}/3 = A_3/24$
J	Jayatilleke function (relative resistance of sublayer to heat and momentum transfer from a smooth wall), Eq (8.5)
K	acceleration parameter, $K \equiv (\nu/U_\infty^2)(dU_\infty/dx)$
K	mean flow kinetic energy, $K \equiv \frac{1}{2}U_i^2$
k	turbulent kinetic energy, $k \equiv \frac{1}{2}\overline{u_i u_i}$
L, $\mathcal{L}$	characteristic flow dimension
L	integral turbulent length scale (usually defined as $k^{3/2}/\varepsilon$; for definitions of bounded length scale in elliptic relaxation models see Eqs (6.74, 7.45))
l	turbulence length scale $k^{3/2}/\varepsilon$
ℓ	alternative turbulence length scale (used in the Wilcox–Rubesin model), $\ell = c_\mu l$
$\mathcal{M}_{ij}$	stress production due to fluctuating (electro)-magnetic (Lorenz) force, Eq (4.95)
N	bulk-flow Stuart number, $N \equiv \sigma B_0^2 L/\rho U_b$
n_i, $\mathbf{n}$	wall-normal unit vector
$\hat{P}$, P, p	instantaneous, mean and fluctuating pressure
P^+	non-dimensional pressure gradient $P^+ = \nu(\partial P/\partial x)/\rho U_\tau^3$
P	wall-adjacent grid node
$\mathcal{P}_{ij}$	stress production due to mean velocity gradient, Eq (2.18)
$\mathcal{P}_{\theta i}$	production of turbulent scalar flux $\overline{\theta u_i}$, Eq (2.22)
$\mathcal{P}_{\theta\theta}$	production of the mean-square scalar variance $\overline{\theta^2}$, Eq (3.20)
$\mathcal{P}_\phi$	production of a turbulence variable ϕ by gradients of mean and fluctuating properties
Pr	molecular Prandtl–Schmidt number
$\dot{q}''_w$	wall heat flux
R	pipe radius
R	thermal-to-mechanical time scale ratio, $\mathsf{R} = \overline{\theta^2}\varepsilon/k\varepsilon_{\theta\theta}$
Ra	Rayleigh number, $\mathrm{Ra} \equiv \beta g(\Theta_w - \Theta_{ref})L^3/\alpha\nu$, where L is a characteristic flow dimension, Θ_w and Θ_{ref} denote the wall and reference temperatures respectively

Re_H	Reynolds number of flow behind a backward-facing step of height H
Re_L	Reynolds number based on a characteristic flow dimension L and velocity U_0, $\mathrm{Re}_L \equiv U_0 L/\nu$
Re_M	magnetic Reynolds number, $\mu_0 \sigma U L$ ($(\mu_0\sigma)^{-1}$ is known as the *magnetic diffusivity*)
Re_m	channel flow Reynolds number based on the mean (bulk) velocity, $\mathrm{Re}_m \equiv U_m 2h/\nu$
Re_t	turbulent Reynolds number, $\mathrm{Re}_t \equiv k^2/(\nu\varepsilon)$
Re_{δ_S}	Reynolds number based on Stokes thickness and maximum free-stream velocity
Re_θ	Reynolds number based on momentum thickness, $\mathrm{Re}_\theta \equiv U_\infty \theta/\nu$
Re_τ	Reynolds number based on friction velocity and channel half-width, $\mathrm{Re}_\tau = U_\tau h/\nu$
Re_λ	Taylor micro-scale Reynolds number, $\mathrm{Re}_\lambda \equiv \sqrt{\overline{u_1^2}}\lambda/\nu$
R_f	flux Richardson number, $-\mathcal{G}_k/\mathcal{P}_k$
Ri	gradient Richardson number, $\mathrm{R}_f \sigma_\Theta$
$R_{ij}(\mathbf{x}, \mathbf{x}')$	two-point correlation tensor, $R_{ij}(\mathbf{x}, \mathbf{x}') \equiv \overline{u_i(\mathbf{x}) u_j(\mathbf{x}')}$
Ro	bulk rotation number (various definitions according to specific application comprising rotating velocity divided by some other reference velocity)
$\mathcal{R}_{ij}$	stress production due to system rotation, Eqs (2.19, 4.68)
r	radial coordinate
r_i, $\mathbf{r}$	position vector
r	mechanical-to-scalar time scale ratio, $r \equiv k\varepsilon_{\theta\theta}/(\overline{\theta^2}\varepsilon) \equiv 1/R$
S	invariant of the non-dimensional mean-strain tensor $\mathrm{S} \equiv \sqrt{\tilde{S}_{mn}\tilde{S}_{nm}}$
S	dimensionless mean strain (in simple shear), $S \equiv 2(k/\varepsilon)(S_{12}S_{12})^{1/2} = (k/\varepsilon)dU/dy$
S	salt concentration ('salinity')
S	invariant of the strain rate tensor $\mathsf{S} \equiv \sqrt{S_{ij}S_{ji}}$
S^*	alternative invariant of mean strain tensor used by Yakhot's group, $\mathrm{S}^* = \sqrt{2}\mathrm{S}$, Eq (5.4)
S_{ij}	mean rate of strain tensor, $S_{ij} \equiv \frac{1}{2}(\partial U_i/\partial x_j + \partial U_j/\partial x_i)$
$\tilde{S}_{ij}$	non-dimensional mean rate of strain, $\tilde{S}_{ij} \equiv S_{ij}k/\varepsilon$
S_w	swirl intensity, a dimensionless ratio of axial to circumferential momentum $S_w \equiv 2\pi \int_0^R UWr^2 dr/\pi R^3 U_b^2$ or $S_w = \int_0^R UWr^2 dr/R \int_0^R U^2 r dr$

$\mathcal{S}_{\varepsilon 1}, \mathcal{S}_{\varepsilon 2}$	general symbols for the source and sink terms in the ε-equation, respectively
s_{ij}	fluctuating rate of strain $s_{ij} \equiv \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$
T	temperature [°K]
$\mathcal{T}$	characteristic turbulence time scale (usually $\mathcal{T}$ is taken as k/ε, but not in Eq (5.23); for definitions of bounded time scale in elliptic relaxation models see Eqs (6.74, 7.35 and 7.44))
$\mathcal{T}(\kappa)$	spectral energy transfer rate
$\mathbb{T}_{ij}^{(n)}$	tensor integrity bases
t	time
U	streamwise mean velocity component
U, V, W	Cartesian components of mean velocity
$\hat{U}_i, U_i, u_i$	instantaneous, mean and fluctuating velocity vector
$U_m, U_b, \bar{U}$	bulk velocity
U_w	wall velocity
U_∞	free stream velocity
U_τ	friction velocity, $\sqrt{\tau_w/\rho}$
U^+	mean velocity non-dimensionalized with friction velocity, $U^+ \equiv U/U_\tau$
U^*	mean velocity for use in wall functions, $U^* \equiv Uk^{1/2}/U_\tau^2 \equiv \rho U k^{1/2}/\tau_w$
ΔU	streamwise velocity change across free shear flow
$-\overline{u_i u_j}$	kinematic Reynolds-stress tensor
u, v, w	Cartesian representation of turbulent velocities
V	mean velocity component in direction y
Va	Valensi number, $\mathrm{Va} \equiv R^2\omega/\nu$
W	invariant of the non-dimensional rotation rate, $\mathrm{W} \equiv \sqrt{\tilde{W}_{ij}\tilde{W}_{ij}}$
W	spanwise and circumferential velocity component
Wo	Womersley number, $\mathrm{Wo} \equiv R\sqrt{\omega/\nu} = \sqrt{\mathrm{Va}}$
W_{wall}	circumferential velocity of rotating wall
W_{ij}	mean rate-of-rotation tensor, $W_{ij} = \frac{1}{2}(\partial U_i/\partial x_j - \partial U_j/\partial x_i)$
$\tilde{W}_{ij}$	non-dimensional mean rate of rotation tensor, $\tilde{W}_{ij} \equiv W_{ij}k/\varepsilon$
w_{ij}	fluctuating rate-of-rotation tensor, $w_{ij} = \frac{1}{2}(\partial u_i/\partial x_j - \partial u_j/\partial x_i)$
x_i, $\mathbf{x}$	Cartesian coordinates in index and vector notation
x, y, z	Cartesian coordinates
y	wall distance
y^+	non-dimensionalized wall distance, $y^+ = U_\tau y/\nu$
y^*	alternative normalized wall-distance, $k^{1/2}y/\nu$
$y_{1/2}$	half-width of plane jet or wake