P.M. Cohn

An Introduction to Ring Theory

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\int_R \nabla \cdot \vec{F}\, dV = \int_{\partial R} \vec{F} \cdot \vec{n}\, d\sigma \longleftrightarrow \int_R dw = \int_{\partial R} w$$

$$\sim(P \cdot Q) \equiv \sim P \vee \sim Q, \sim(P \vee Q) \equiv \sim P \cdot \sim Q$$

$$\left|\left\langle \chi, \gamma \right\rangle\right| \leq \left\|\chi\right\| \left\|\gamma\right\|$$

$$\delta_{ij} = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \frac{1}{|G|} \sum_{l=1} k_l \chi_i(g_l)\overline{\chi_j(g_l)}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

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For my grandchildren
James, Olivia and Hugo Aaronson

Preface

The theory of rings, a newcomer on the algebra scene in the late 19th century, has grown in importance throughout the 20th century, and is now a well-established part of algebra. This introduction to the subject is intended for second and third year students and postgraduates, but would also be suitable for self-study. The reader is assumed to have met sets, groups and vector spaces, but any results used are clearly stated, with a reference to sources.

After a chapter on the basic definitions of rings, modules and a few categorical notions, there is a chapter each on Artinian rings and on (commutative) Noetherian rings, followed by a presentation of some important constructions and concepts in ring theory, such as direct products, tensor products and the use of projective and injective modules. Of the many possible applications we have chosen to include at least a brief account of representation theory and algebraic number theory.

In the final chapter an author is usually allowed to indulge his personal tastes; here it is devoted to free algebras and more generally, free ideal rings. These are topics that have only recently been developed, whose basic theory is relatively simple, and which are not too widely known, although they are now beginning to find applications in topology and functional analysis.

Care has been taken to present all proofs in detail, while including motivation and providing material for the reader to practise on. Most results are accompanied by worked examples and each section has a number of exercises, for whose solution some hints are given at the end of the book.

I would like to express my thanks to the Publishers, to their Mathematics Editor Mr. D. Ireland and especially to their Copy Editor Mr S.M. Nugent for the expert way in which he has helped to tame the manuscript. I would also like to thank an anonymous reviewer for bringing a number of mistakes in the original manuscript to my attention.

September 1999 P. M. Cohn

Contents

Introduction

Most parts of algebra have undergone great changes in the course of the 20th century; probably none more so than the theory of rings. While groups were well established early in the 19th century, in its closing years the term "ring" just meant a "ring of algebraic integers" and the axiomatic foundations were not laid until 1914 (Fraenkel 1914). In the early years of the 20th century Wedderburn proved his theorems about finite-dimensional algebras, which some time later were recognized by Emmy Noether to be true of rings with chain condition; Emil Artin realized that the minimum condition was enough and such rings are now called Artinian. Both E. Noether and E. Artin lectured on algebra in Göttingen in the 1920s and B.L. van der Waerden, who had attended these lectures, decided to write a textbook based on them which first came out in 1931 and rapidly became a classic. Significantly enough it was entitled "Moderne Algebra" on its first appearance, but the "Modern" was dropped from the title after 25 years.

Meanwhile in commutative ring theory the properties of rings of algebraic numbers and coordinate rings of algebraic varieties were established by E. Noether and others for more general rings: apart from commutativity only the maximum condition was needed and the rings became known as Noetherian rings.

Whereas the trend in mathematics at the beginning of the 20th century was towards axiomatics, the emphasis since the second world war has been on generality, where possible, regarding several subjects from a single point of view. Thus the algebraicization of topology has led to the development of homological algebra, which in turn has had a major influence on ring theory.

In another field, the remarkable progress in algebraic geometry by the Italian School has been put on a firm algebraic basis, and this has led to progress in commutative ring theory, culminating in the result which associates with any commutative ring an affine scheme. Thirdly the theory of operator algebras, which itself received an impetus from quantum mechanics, has led to the development of function algebras. More recently the study of "non-commutative geometry" has emphasized the role of non-commutative rings.

To do justice to all these trends would require several volumes and many authors. The present volume merely takes the first steps in such a development, introducing the definition and basic notions of rings and constructions, such as rings of fractions, residue-class rings and matrix rings, to be followed by brief accounts of Artinian rings, commutative Noetherian rings, and a chapter on free rings.

Suggestions for further reading are included in the Bibliography at the end of the book. References to the author's three-volume work on Algebra are abbreviated to A.1, A.2, and A.3, when cited in the text.

Remarks on Notation and Terminology

Most of the notation will be explained when it first occurs, but we assume that such ideas as the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, and real numbers $\mathbb{R}$, are known, as well as the membership sign: $\in$, as in $3 \in \mathbb{N}$, inclusion $\subseteq$ as in $\mathbb{N} \subseteq \mathbb{Z}$, or even $\mathbb{N} \subset \mathbb{Z}$, where "$\subset$" denotes proper inclusion; one also writes $\mathbb{Z} \supseteq \mathbb{N}$, $\mathbb{Z} \supset \mathbb{N}$. This notation for the various systems was introduced by Bourbaki in the late 1930s and is now used worldwide; until that time, surprisingly, no universally used names existed. Instead of $\mathbb{N}$ one sometimes writes $\mathbb{Z}^+$ (the integers ≥ 0). Similarly, $\mathbb{Q}^+$, $\mathbb{R}^+$ indicates the non-negative rational, resp., real numbers. Since every non-zero real number has a positive square, equations such as $x^2 + 1 = 0$ have no real solution, but by introducing a solution i, one obtains the set of all complex numbers $x + y\mathrm{i}(x, y$ real), usually denoted by $\mathbb{C}$. Then one can show that every polynomial equation with complex coefficients has a complex root, a fact expressed by saying that $\mathbb{C}$ is **algebraically closed.**

Of course we assume that the reader is familiar with the notion of a **set**, as a collection of objects, called **elements** or **members** of the set. Given sets S, T, their intersection is written $S \cap T$, their union is $S \cup T$, and their direct product, consisting of all pairs (x, y), where $x \in S, y \in T$, is $S \times T$. Sometimes a slight variant is needed: suppose we have a relation between vectors x_1, x_2, x_3 in a vector space, e.g. $a_1x_1 + a_2x_2 + a_3x_3 = 0$; we may wish to consider the numbers a_1, a_2, a_3, but if $a_1 = a_2$ say, the set $\{a_1, a_2, a_3\}$ will reduce to $\{a_1, a_3\}$ and so will not distinguish between a_1 and a_2. Instead we shall speak of the **family** (a_1, a_2, a_3), **indexed** by the set $\{1, 2, 3\}$.

As a rule we shall use Greek letters for general indexing sets, and Latin letters for finite sets. Sometimes a set with no elements is considered; this is the **empty set**, denoted by $\varnothing$; this notation was introduced by Lefschetz in 1942 (Solomon Lefschetz, 1884–1972). A property which holds for all except a finite number of members of a set S is said to hold for **almost all** members of S. If S' is a subset of a set S, its **complement,** i.e. the set $\{x \in S | x \notin S'\}$, is denoted by $S \setminus S'$.

Our readers will also have met groups before, but for completeness we include their definition here. By a **group** we understand a non-empty set G with a binary operation, called **multiplication,** associating with each pair of elements a, b of G another element of G, written $a.b$ or ab and called the **product** of a and b, such that:

G.1 $a(bc) = (ab)c$; (associative law)

G.2 for any $a, b \in G$ the equations $ax = b, ya = b$ each have a solution.

It turns out that the solutions x, y in G.2 are uniquely determined by a and b. The element e satisfying $ae = a$ also satisfies $ea = a$ for all $a \in G$ and is called the **unit element** or **neutral element** of G. If only G.1 holds, we have a set with an associative multiplication; this is called a **semigroup.** A semigroup with a neutral element is called a **monoid**.

Most of our sets will be finite; those that are infinite will usually be **countable**, i.e. the elements can be enumerated by means of the natural numbers $\mathbb{N}$. No doubt all readers are aware that there are uncountable sets (e.g. the real numbers $\mathbb{R}$), but their properties will not concern us here, though in Chapter 4 we shall briefly deal with this more general situation.

We shall frequently use the notion of a partial ordering on a set and so briefly recall the definition. A **partially ordered** set is a set S with a binary relation $\leq$ satisfying the following conditions:

O.1 for all $x \in S, x \leq x$ (reflexivity);

O.2 for all $x, y, z \in S, x \leq y, y \leq z$ implies $x \leq z$ (transitivity);

O.3 for all $x, y \in S, x \leq y, y \leq x$ implies $x = y$ (antisymmetry).

We shall also write "$x < y$" for "$x \leq y$ and $x \neq y$" and instead of "$x \leq y$" sometimes write "$y \geq x$"; similarly "$x > y$" stands for "$y < x$". If in addition, we have

O.4 for all $x, y \in S, x \leq y$ or $y \leq x$,

the ordering is said to be **total** or **linear**. In any partially ordered set a totally ordered subset is called a **chain**.

Ordered sets abound in mathematics; for example, the natural numbers form a set which is totally ordered by size and partially (but not totally) ordered by