Infinite dimensional Lie algebras

THIRD EDITION

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Dedicated to my teacher,

Ernest Borisovich Vinberg

with gratitude and admiration

La Nature est un temple où de vivants piliers
Laissent parfois sortir de confuses paroles;
L'homme y passe à travers des forêts de symboles
Qui l'observent avec des regards familiers.
Charles Baudelaire, *Les Fleurs du Mal*

И я выхожу из пространства
В запущенный сад величин
Osip Mandelstam

Introduction

§0.1. The creators of the Lie theory viewed a Lie group as a group of symmetries of an algebraic or a geometric object; the corresponding Lie algebra, from their point of view, was the set of infinitesimal transformations. Since the group of symmetries of the object is not necessarily finite-dimensional, S. Lie considered not only the problem of classification of subgroups of GL_n, but also the problem of classification of infinite-dimensional groups of transformations.

The problem of classification of simple finite-dimensional Lie algebras over the field of complex numbers was solved by the end of the 19th century by W. Killing and E. Cartan. (A vivid description of the history of this discovery, one of the most remarkable in all of mathematics, can be found in Hawkins [1982].) And just over a decade later, Cartan classified simple infinite-dimensional Lie algebras of vector fields on a finite-dimensional space.

Starting with the works of Lie, Killing, and Cartan, the theory of finite-dimensional Lie groups and Lie algebras has developed systematically in depth and scope. On the other hand, Cartan's works on simple infinite-dimensional Lie algebras had been virtually forgotten until the mid-sixties. A resurgence of interest in this area began with the work of Guillemin–Sternberg [1964] and Singer–Sternberg [1965], which developed an adequate algebraic language and the machinery of filtered and graded Lie algebras. They were, however, unable to find an algebraic proof of Cartan's classification theorem (see Guillemin–Quillen–Sternberg [1966] for an analytic proof). This was done by Weisfeiler [1968], who reduced the problem to the classification of simple $\mathbb{Z}$-graded Lie algebras of finite "depth" $\mathfrak{g} = \bigoplus_{j \geq -d} \mathfrak{g}_j$, where dim $\mathfrak{g}_j < \infty$ and the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible.

§0.2. At the present time there is no general theory of infinite-dimensional Lie groups and algebras and their representations. There are, however, four classes of infinite-dimensional Lie groups and algebras that underwent a more or less intensive study. These are, first of all, the above-mentioned Lie algebras of vector fields and the corresponding groups of diffeomorphisms of a manifold. Starting with the works of Gelfand–Fuchs [1969], [1970A,B], there emerged an important direction having many geometric applications, which is the cohomology theory of infinite-dimensional Lie algebras of vector fields on a finite-dimensional manifold. There is also a rather large number of works which study and classify various classes of representations of the groups of diffeomorphisms of a manifold. One should probably include in the first class the groups of biregular automorphisms of an algebraic variety (see Shafarevich [1981]).

The second class consists of Lie groups (resp. Lie algebras) of smooth mappings of a given manifold into a finite-dimensional Lie group (resp. Lie algebra). In other words, this is a group (resp. Lie algebra) of matrices over some function algebra but viewed over the base field. (The physicists refer to certain central extensions of these Lie algebras as current algebras.) The main subject of study in this case has been certain special families of representations.

The third class consists of the classical Lie groups and algebras of operators in a Hilbert or Banach space. There is a rather large number of scattered results in this area, which study the structure of these Lie groups and algebras and their representations. A representation which plays an important role in quantum field theory is the Segal–Shale–Weil (or metaplectic) representation of an infinite-dimensional symplectic group.

I shall not discuss in this book the three classes of infinite-dimensional Lie algebras listed above, with the exception of those closely related to the Lie algebras of the fourth class, which we consider below. The reader interested in these three classes should consult the literature cited at the end of the book.

Finally, the fourth class of infinite-dimensional Lie algebras is the class of the so-called Kac–Moody algebras, the subject of the present book.

§0.3. Let us briefly discuss the main concepts of the structural theory of Kac–Moody algebras. Let $A = (a_{ij})_{i,j=1}^n$ be a *generalized Cartan matrix*, i.e., an integral $n \times n$ matrix such that $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$, and $a_{ij} = 0$

implies $a_{ji} = 0$. The associated *Kac–Moody algebra* $\mathfrak{g}'(A)$ is a complex Lie algebra on $3n$ generators e_i, f_i, h_i ($i = 1, \dots, n$) and the following defining relations ($i, j, = 1, \dots, n$):

$$(0.3.1) \qquad \begin{cases} [h_i, h_j] = 0, \quad [e_i, f_i] = h_i, \quad [e_i, f_j] = 0 \text{ if } i \neq j, \\ [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \\ (\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \quad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0 \text{ if } i \neq j. \end{cases}$$

(The definition given in the main text of the book (see Chapter 1) is different from the above; it is more convenient for a number of reasons. The proof of the fact that the derived algebra of the Lie algebra $\mathfrak{g}(A)$ defined in Chapter 1 coincides with the Lie algebra $\mathfrak{g}'(A)$ defined by relations (0.3.1) has been obtained by Gabber–Kac [1981] under a "symmetrizability" assumption; this proof appears in Chapter 9.)

I came to consider these Lie algebras while trying to understand and generalize the works of Guillemin–Quillen–Singer–Sternberg–Weisfeiler on Cartan's classification. The key idea was to consider arbitrary simple $\mathbb{Z}$-graded Lie algebras $\mathfrak{g} = \bigoplus_j \mathfrak{g}_j$; but since there are too many such Lie algebras, the point was to require the dimension of $\mathfrak{g}_j$ to grow no faster than some polynomial in j. (One can show that Lie algebras of finite depth do satisfy this condition, and that this condition is independent of the gradation.) Such Lie algebras were classified under some technical hypotheses (see Kac [1968 B]). It turned out that in addition to Cartan's four series of Lie algebras of polynomial vector fields, there is another class of infinite-dimensional Lie algebras of polynomial growth, which are called affine Lie algebras (more precisely, they are the quotients of affine Lie algebras by the 1-dimensional center). At the same time, Moody [1968] independently undertook the study of the Lie algebras $\mathfrak{g}'(A)$.

The class of Kac–Moody algebras breaks up into three subclasses. To describe them, it is convenient to assume that the matrix A is *indecomposable* (i.e., there is no partition of the set $\{1, \dots, n\}$ into two nonempty subsets so that $a_{ij} = 0$ whenever i belongs to the first subset, while j belongs to the second; this is done without loss of generality since the direct sum of matrices corresponds to the direct sum of Kac–Moody algebras). Then there are the following three mutually exclusive possibilities:

a) There is a vector θ of positive integers such that all the coordinates of the vector $A\theta$ are positive. In such case all the principal minors of the matrix A are positive and the Lie algebra $\mathfrak{g}'(A)$ is finite-dimensional.

b) There is a vector δ of positive integers such that $A\delta = 0$. In such case all the principal minors of the matrix A are nonnegative and $\det A = 0$; the algebra $\mathfrak{g}'(A)$ is infinite-dimensional, but is of polynomial growth (moreover, it admits a $\mathbb{Z}$-gradation by subspaces of uniformly bounded dimension). The Lie algebras of this subclass are called *affine Lie algebras.*

c) There is a vector α of positive integers such that all the coordinates of the vector $A\alpha$ are negative. In such case the Lie algebra $\mathfrak{g}'(A)$ is of exponential growth.

The main achievement of the Killing–Cartan theory may be formulated as follows: a simple finite-dimensional complex Lie algebra is isomorphic to one of the Lie algebras of the subclass a). (Note that the classification of matrices of type a) and b) is a rather simple problem.) The existence of the generators satisfying relations (0.3.1) was pointed out by Chevalley [1948] and Harish–Chandra [1951]. (Much later Serre [1966] and Kac [1968 B] showed that these are defining relations.)

It turned out that most of the classical concepts of the Killing–Cartan–Weyl theory can be carried over to the entire class of Kac–Moody algebras, such as the Cartan subalgebra, the root system, the Weyl group, etc. In doing so one discovers a series of new phenomena, which the book treats in detail (see Chapters 1–6). I shall only point out here that $\mathfrak{g}'(A)$ does not always possess a nonzero invariant bilinear form. This is the case if and only if the matrix A is *symmetrizable*, i.e., the matrix DA is symmetric for some invertible diagonal matrix D (see Chapter 2).

§0.4. It is an important property of affine Lie algebras that they possess a simple realization (see Chapters 7 and 8). Here I shall explain this realization for the example of the Kac–Moody algebra associated to the extended Cartan matrix A of a simple finite-dimensional complex Lie algebra $\mathfrak{g}$. (All such matrices are "affine" generalized Cartan matrices; the corresponding algebra $\mathfrak{g}'(A)$ is called a nontwisted affine Lie algebra.) Namely, the affine Lie algebra $\mathfrak{g}'(A)$ is a central extension by the 1-dimensional center of the Lie algebra of polynomial maps of the circle into the simple finite-dimensional complex Lie algebra $\mathfrak{g}$ (so that it is the simplest example of a Lie algebra of the second class mentioned in §0.2).

More precisely, let us consider the Lie algebra $\mathfrak{g}$ in some faithful finite-dimensional representation. Then the Lie algebra $\mathfrak{g}'(A)$ is isomorphic to the Lie algebra on the complex space $(\mathbb{C}[t,t^{-1}]\otimes_{\mathbb{C}}\mathfrak{g})\oplus\mathbb{C}c$ with the bracket

$$[(t^m\otimes a)\oplus\lambda c,(t^n\otimes b)\oplus\mu c]=\left(t^{m+n}\otimes[a,b]\right)\oplus m\delta_{m,-n}(\mathrm{tr}\,ab)c,$$

so that $\mathbb{C}c$ is the (1-dimensional) center. This realization allows us to study affine Lie algebras from another point of view. In particular, the algebra of vector fields on the circle (the simplest algebra of the first class) plays an important role in the theory of affine Lie algebras.

Note also that the Lie algebras of the fourth class are closely related to the affine Lie algebras of infinite rank, considered in Chapters 7 and 14.

Unfortunately, no simple realization has been found up to now for any nonaffine infinite-dimensional Kac–Moody algebra. This question appears to be one of the most important open problems of the theory.

§0.5. An important concept missing from the first works in Kac–Moody algebras was the concept of an integrable highest-weight representation (introduced in Kac [1974]). Given a sequence of nonnegative integers $\Lambda = (\lambda_1, \ldots, \lambda_n)$, the ***integrable highest-weight representation*** of a Kac–Moody algebra $\mathfrak{g}'(A)$ is an irreducible representation π_Λ of $\mathfrak{g}'(A)$ on a complex vector space $L(\Lambda)$, which is determined by the property that there is a nonzero vector $v_\Lambda \in L(\Lambda)$ such that

$$\pi_\Lambda(e_i)v_\Lambda = 0 \text{ and } \pi_\Lambda(h_i)v_\Lambda = \lambda_i v_\Lambda \quad (i = 1, \ldots, n).$$

(This terminology is explained by the fact that Λ is called the highest-weight, and the conditions on Λ are necessary and sufficient for being able to integrate π_Λ and obtain a representation of the group.)

Cartan's theorem on the highest-weight asserts that all the representations π_Λ of a complex simple finite-dimensional Lie algebra are finite-dimensional, and that every finite-dimensional irreducible representation is equivalent to one of the π_Λ.

That the representations π_Λ are finite-dimensional (the most nontrivial part of Cartan's theorem) was proved by Cartan by examining the cases, one by one. A purely algebraic proof was found much later by C. Chevalley [1948] and Harish and Chandra [1951] (a "transcendental" proof had been found earlier by H. Weyl). This brief note by Chevalley appears in retrospect as the precursor of the algebraization of the representation theory of Lie groups. This note also contains, in an embryonic form, many of the basic concepts of the theory of Kac–Moody algebras.

The algebraization of the representation theory of Lie groups, which has undergone such an explosive development during the last decade, started with the work Bernstein–Gelfand–Gelfand [1971] on Verma modules (the first nontrivial results about these modules were obtained by Verma [1968]). In particular, using the Verma modules, Bernstein–Gelfand–Gelfand gave a transparent algebraic proof of Weyl's formula for the characters of finite-dimensional irreducible representations of finite-dimensional simple Lie algebras.

At about the same time Macdonald [1972] obtained his remarkable identities. In this work he undertook to generalize the Weyl denominator identity to the case of affine root systems. He remarked that a straightforward generalization is actually false. To salvage the situation he had to add some "mysterious" factors, which he was able to determine as a result of lengthy calculations. The simplest example of Macdonald's identities is the famous Jacobi triple product identity:

$$\prod_{n\geq 1}(1-u^n v^n)(1-u^{n+1}v^n)(1-u^n v^{n+1}) = \sum_{n\in\mathbb{Z}}(-1)^m u^{\frac{1}{2}m(m+1)}v^{\frac{1}{2}m(m-1)}.$$

The "mysterious" factors which do not correspond to affine roots are the factors $(1-u^n v^n)$.

After the appearance of the two works mentioned above very little remained to be done: one had to place them on the desk next to one another to understand that Macdonald's result is only the tip of the iceberg—the representation theory of Kac–Moody algebras. Namely, it turned out that a simplified version of Bernstein–Gelfand–Gelfand's proof may be applied to the proof of a formula generalizing Weyl's formula, for the formal character of the representation π_Λ of an arbitrary Kac–Moody algebra $\mathfrak{g}'(A)$ corresponding to a symmetrizable matrix A. In the case of the simplest 1-dimensional representation π_0, this formula becomes the generalization of Weyl's denominator identity. In the case of an affine Lie algebra, the generalized Weyl denominator identity turns out to be equivalent to the Macdonald identities. In the process, the "mysterious" factors receive a simple interpretation: they correspond to the so-called imaginary roots (i.e., roots that one should add to the affine roots to obtain all the roots of the affine Lie algebra). Note that the simplest example of the Jacobi triple product identity turns out to be just the generalized denominator identity for the affine Lie algebra corresponding to the matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$.

The exposition of these results (obtained by Kac [1974]) may be found in Chapter 10. Chapters 9–14 are devoted to the general theory of highest-weight representations and their applications.

The main tool of the theory of representations with highest-weight is the generalized Casimir operator (see Chapter 2). Unfortunately, the construction of this operator depends on whether the matrix A is symmetrizable. The question whether one can lift the hypothesis of symmetrizability of the matrix A remains open.

Once the integrable highest-weight representations had been introduced, the theory of Kac–Moody algebras got off the ground and has been developing since at an accelerating speed. In the past decade this theory has emerged as a field that has close connections with many areas of mathematics and mathematical physics, such as invariant theory, combinatorics, topology, the theory of modular forms and theta functions, the theory of singularities, finite simple groups, Hamiltonian mechanics, soliton equations, and quantum field theory.

§0.6. This book contains a detailed exposition of the foundations of the theory of Kac–Moody algebras and their integrable representations. Besides the application to the Macdonald identities mentioned above (Chapter 12), the book discusses the application to the classification of finite-order automorphisms of simple finite-dimensional Lie algebras (Chapter 8), and the connection with the theory of modular forms and theta functions (Chapter 13). The last chapter (Chapter 14) discusses the remarkable connection between the representation theory of affine Lie algebras and the Korteweg–de Vries-type equations, discovered by the Kyoto school.

A theory of Lie algebras is usually interesting, insofar as it is related to group theory, and Kac–Moody algebras are no exception. Recently there appeared a series of deep results on groups associated with Kac–Moody algebras. A discussion of these results would require writing another book. I chose to make only a few comments regarding this subject at the end of some chapters.

§0.7. Throughout the book the base field is the field of complex numbers. However, all the results of the book, except, of course, for the ones concerning Hermitian forms and convergence problems, can be extended without difficulty to the case of an arbitrary field of characteristic zero.

§0.8. Motivations are provided at the beginning of each chapter, which ends with related bibliographical comments. The main text of each chapter is followed by exercises (whose total number exceeds 250). Some of them are elementary, others constitute a brief exposition of original works. I hope that these expositions are sufficiently detailed for the diligent reader to reconstruct all the proofs. The square brackets at the end of some exercises contain hints for their solution.

The exposition in the book is practically self-contained. Although I had in mind a reader familiar with the theory of finite-dimensional semisimple Lie algebras, what would suffice for the most part is a knowledge of the elements of Lie algebras, their enveloping algebras and representations. For example, the book of Humphreys [1972] or Varadarajan [1984] is more than sufficient.

One finds a rather extensive bibliography at the end of the book. I hope that the collection of references to mathematical works in the theory of Kac–Moody algebras is at least everywhere dense. This is not at all so in the case of the works in physics. The choice of references in this case was rather arbitrary and often depended on whether I had a copy of the paper or discussed it with the author. The same should be said as regards the references to the works on the other classes of infinite-dimensional Lie algebras.

§0.9. This book is based on lectures given at MIT in 1978, 1980, and 1982, and at the Collège de France in 1981. I would like to thank those who attended for helpful comments and corrections of the notes, in particular F. Arnold, R. Coley, R. Gross, Z. Haddad, M. Haiman, G. Heckman, F. Levstein, A. Rocha, and T. Vongiouklis. I am grateful to M. Duflo, G. Heckman, B. Kupershmidt, and B. Weisfeiler for reading some parts of the manuscript and pointing out errors. I apologize for those errors that remain. My thanks go to F. Rose, B. Katz, and M. Katz without whose help and support this book would never have come out. I also owe thanks to K. Manning and C. Macpherson for help with the language. The book was prepared using D. Knuth's TEX. Finally, I would like on this occasion to express my deep gratitude to D. Peterson, whose collaboration had a great influence not only on this book, but also on most of my mathematical work in the past few years.

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July 1983, L'Isle Adam, France.

Preface to the Second Edition

The most important additions reflect recent developments in the theory of infinite-dimensional groups (some key facts, like Proposition 3.8 and Exercise 5.19 are among them) and in the soliton theory (like Exercises 14.37–14.40 which uncover the role of the Virasoro algebra). The most important correction concerns the proof of the complete reducibility Proposition 9.10. The previous proof used Lemma 9.10 b) of the first edition which is false, as Exercise 9.15 shows. A correct version of Lemma 9.10 b) is the new Proposition 10.4 which gives a characterization of integrable highest-weight modules.

In addition to correcting misprints and errors and adding a few dozen of new exercises, I have brought to date the list of references and related bibliographical comments. I want to thank those who have pointed out errors and suggested improvements, in particular: J. Dorfmeister, T. Enright, D. Freed, E. Getzler, E. Gutkin, P. de la Harpe, S. Kumar, B. Kupershmidt, S.-R. Lu, D. Peterson, L.-J. Santharoubane, G. Schwarz, V S. Varadarajan, M. Wakimoto, Z.-X. Wan, X.-D. Wang, Y.-X. Wang, B. Weisfeiler, C.-F. Xie, Y.-C. You, H.-C. Zhang.

April 1985, Cambridge, Massachusetts.

Preface to the Third Edition

This edition differs considerably from the previous ones. Particularly, more emphasis is made on connections to mathematical physics, especially to conformal field theory.

Below is a list of the most important improvements and additions:

Chapter 3. A simplest example of a quantized Kac–Moody algebra, $U_q(sl_2)$, is given, along with its representations (Exercises 3.23 and 3.34).

Chapter 5. The hyperbolic Weyl group theory is applied to the study of the unimodular Lorentzian lattices of rank ≤ 10 (§5.10).

Chapter 6. An explicit construction of all finite type root and coroot lattices is given, along with the associated Weyl group, root systems, etc. (§6.7).

Chapter 7. The field theoretic approach to affine algebras is briefly outlined (§7.7). An explicit construction of all simple finite-dimensional Lie algebras is given in terms of the root lattice and an "asymmetry function" on it (§§7.8–7.10).

Chapter 8. A simple and self-contained proof is given of the basic fact about twisted affine algebras: the equivariant loop algebra $\mathcal{L}(\mathfrak{g}, \sigma, m)$ depends only on the connected component of Aut $\mathfrak{g}$ containing σ (§8.5).

Chapter 9. Elements of the representation theory of the Virasoro algebras are discussed (§9.14). A free field construction of representations of the Virasoro algebra and the affine algebra of type $A_1^{(1)}$ is given (Exercises 9.17–9.20).

Chapter 11. Unitarizability of representations of the Virasoro algebra is discussed (§11.12). A theory of generalized Kac–Moody algebras is outlined (§11.13).

Chapter 12. The Sugawara construction and the coset construction, which are the basic constructions of conformal field theory, are explained. The general branching functions and vacuum pairs are introduced in this context (§§12.8–12.13).

Chapter 13. General branching functions are studied along with string functions. The matrix S of the modular transformations of characters is studied (this was implicit in earlier editions). Explicit estimates of the orders of all poles and of levels of branching functions are given. Asymptotics of characters and branching functions at high temperature limit is studied, along with the related positivity conjecture. The interplay between the modular and conformal invariance constraints is demonstrated (§§13.8–13.14). This is used to study unitarizable representations of the Virasoro algebra, and to calculate the fusion rules (Exercises 13.18–13.26, and 13.34–13.36).

Chapter 14. The homogeneous vertex operator construction is derived via the vertex operator calculus (§14.8). The infinite wedge representation is constructed (§14.9). By making use of the boson-fermion correspondence (§14.10) the whole KP hierarchy is studied (§§14.11 and 14.12). By making use of the principal and homogeneous vertex operator constructions of $A_1^{(1)}$, the whole KdV and NLS hierarchies are described (§14.13). The BKP hierarchy is constructed (Exercises 14.13–14.15). A theory of the infinite Grassmannian and flag manifold is sketched and their connection to the KP and MKP hierarchies is explained (Exercises 14.32, 14.33). A pseudodifferential operator approach to the KP and KdV hierarchies is outlined (Exercises 14.44–14.51). A basis free theory of the Lie algebra and group of type A_∞ is discussed (Exercises 14.55–14.58), and some classical theorems of the theory of algebraic curves are derived from this discussion (Exercises 14.59–14.63).

In addition to correcting misprints and errors and adding some hundred new exercises, I have brought up to date the list of references and related bibliographical comments. The explosion of activity in the field between the second and the third editions, due to a great extent to physicists working in string theory and conformal field theory, made it an impossible task to compile a reasonably complete bibliography. I hope, however, that the collection of references compiled for this edition at least reflects all the major directions of research in the field. Needless to say that every sentence of my bibliographical comments could be prefixed by an "It is my opinion that ...."

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September 1989, Newton, Massachusetts.

Notational Conventions

$\mathbf{Z}$	the set of integers
$\mathbf{Z}_+$	the set of non-negative integers
$\mathbf{N}$	the set of positive integers
$\mathbf{Q}$	the set of rational numbers
$\mathbf{R}$	the set of real numbers
$\mathbf{R}_+$	the set of non-negative real numbers
$\mathbf{C}$	the set of complex numbers
$S^\times$	the set of invertible elements of a ring S
$\operatorname{Re} z$ and $\operatorname{Im} z$	real and imaginary parts of $z \in \mathbf{C}$
$\log z$	for $z \in \mathbf{C}^\times : e^{\log z} = z$ and $-\pi \leq \operatorname{Im} \log z < \pi$
z^α	$= e^{\alpha \log z}$ for $\alpha \in \mathbf{C}$, $z \in \mathbf{C}^\times$
$U \oplus V$ or $\bigoplus_\alpha U_\alpha$	direct sum of vector spaces
$\sum_\alpha U_\alpha$	sum of subspaces of a vector space
$\prod_\alpha U_\alpha$	direct product of vector spaces
kS	the linear k-span of S ($k = \mathbf{Z}, \mathbf{Z}_+, \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$)
$U \otimes V$	tensor product of vector k-spaces over k ($k = \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$)
$U*$	the dual of a vector k-space over k ($k = \mathbf{Q}, \mathbf{R}$, or $\mathbf{C}$)
k^n	direct sum of n copies of the vector space k ($n \in \mathbf{Z}_+ \cup \{\infty\}$)
I_V or I_n or I	the identity operator on the n-dimensional vector space V
$\langle \cdot , \cdot \rangle$	pairing between a vector space and its dual
$\|u\|^2 = (u\|u)$	square length of a vector u
$\|S\|$	cardinality of a set S
$P \mod Q$	a set of representatives of cosets of an abelian group P with respect to a subgroup Q