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# UNCONDITIONAL CONVERGENCE OF HIGH-ORDER EXTRAPOLATIONS OF THE CRANK-NICOLSON, FINITE ELEMENT METHOD FOR THE NAVIER-STOKES EQUATIONS

ROSS INGRAM

**Abstract.** Error estimates for the Crank-Nicolson in time, Finite Element in space (CNFE) discretization of the Navier-Stokes equations require application of the discrete Gronwall inequality, which leads to a time-step ( $\Delta t$ ) restriction. All known convergence analyses of the fully discrete CNFE with linear extrapolation rely on a similar  $\Delta t$ -restriction. We show that CNFE with arbitrary-order extrapolation (denoted CNLE) is *convergent optimally in the energy norm* without any  $\Delta t$ -restriction. We prove that CNLE velocity and corresponding discrete time-derivative converge optimally in  $l^\infty(H^1)$  and  $l^2(L^2)$  respectively under the mild condition  $\Delta t \leq Mh^{1/4}$  for any arbitrary  $M > 0$  (e.g. independent of problem data,  $h$ , and  $\Delta t$ ) where  $h > 0$  is the maximum mesh element diameter. Convergence in these higher order norms is needed to prove convergence estimates for pressure and the drag/lift force a fluid exerts on an obstacle. Our analysis exploits the extrapolated convective velocity to avoid any  $\Delta t$ -restriction for convergence in the energy norm. However, the coupling between the extrapolated convecting velocity of usual CNLE and the *a priori* control of *average* velocities (characteristic of CN methods) rather than pointwise velocities (e.g. backward-Euler methods) in  $l^2(H^1)$  is precisely the source of  $\Delta t$ -restriction for convergence in higher-order norms.

**Key words.** Navier-Stokes, Crank-Nicolson, finite element, extrapolation, linearization, error, convergence, linearization

## 1. Introduction

The usual Crank-Nicolson (CN) in time Finite Element (FE) in space discretization of the Navier-Stokes (NS) Equations (NSE) denoted by CNFE is well-known to be unconditionally (energetically) stable. The error analysis of the CNFE method is based on a discrete Gronwall inequality which introduces a time-step ( $\Delta t > 0$ ) restriction (for convergence, not for stability) of the form

$$\Delta t \leq C(Re, h), \quad \text{e.g. } \Delta t \leq \mathcal{O}(Re^{-3}) \quad (1)$$

(see Appendix A for a derivation, Theorem A.1 with e.g. (157)). Here  $h > 0$  is the maximum mesh element diameter and  $Re > 0$  is the Reynolds number. Condition (1)(a) implies *conditional convergence* whereas (1)(b) is a *robustness condition* and both are prohibitively restrictive in practice; for example, (1)(b) suggests

$$Re = 100 \text{ (low-to-moderate value)} \Rightarrow \Delta t \leq \mathcal{O}(10^{-6})$$

Consequently, an important open question regards whether condition (1) is

- an artifact of imperfect mathematical technique or
- a special feature of the CN time discretization

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We consider the necessity of a  $\Delta t$ -restriction in a linear, fully implicit variant of CNFE obtained by extrapolation of the convecting velocity  $\mathbf{u}$ : for example, suppressing spatial discretization, given  $\mathbf{u}^0$ ,  $\mathbf{u}^1$ , and  $p^1$ , for each  $n = 1, 2, \dots$  find velocity  $\mathbf{u}^{n+1}$  and pressure  $p^{n+1}$  satisfying

$$\begin{aligned} \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \left(\frac{3}{2}\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1}\right) \cdot \nabla \frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} \\ - Re^{-1} \Delta \frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} + \nabla \frac{p^{n+1} + p^n}{2} = \frac{\mathbf{f}^{n+1} + \mathbf{f}^n}{2} \\ \nabla \cdot \mathbf{u}^{n+1} = 0. \end{aligned} \quad (2)$$

Here  $\mathbf{f}$  is body-force term, and  $z^i := z(x, t^i)$  and  $t^i = i\Delta t$ . This method is often called CNLE and was first studied by Baker [3]. CNLE is linearly implicit, unconditionally (energetically) stable, and second-order accurate. In this report, we show that *no  $\Delta t$ -restriction* is required for the convergence of CNLE (Proposition 3.1, Theorem 3.5). In particular,

$$\|error(CNLE)\|_{l^\infty(L^2) \cap l^2(H^1)} \leq C(h^k + \Delta t^2), \quad k = \text{degree of FE space}$$

(Theorem 3.5). This result was proved for the semi-discrete case as  $\Delta t \rightarrow 0$  in [10] and the fully discrete Backward Euler (BE) scheme with Constant Extrapolation (BECE) in [32]. The analysis depends on

- *Gronwall inequality* - exploit time-lagged convecting velocity (Section 1.1)
- Estimate (74) - bound convecting velocity in  $L^2$  (Section 1.1.1)

Indeed, for extrapolated CN, we apply a discrete Gronwall Lemma without any  $\Delta t$ -restriction; for general extrapolations we derive and apply the estimate (74)(b) of the *explicitly* skew-symmetric convective term. We explain our strategy for proving the CNLE error estimate and corresponding difficulties in detail in Section 1.1.

We also prove convergence estimates in higher-order norms. We show that the CNLE velocity approximation converges optimally in the  $l^\infty(H^1)$ -norm and the corresponding discrete time-derivative of the velocity approximation converges optimally in the  $l^2(L^2)$ -norm (Theorems 3.8, 3.10) under a modest  $\Delta t$ -restriction

$$\Delta t \leq Mh^{1/4}, \quad \text{for any } M > 0 \text{ (no } Re\text{-dependence)}. \quad (3)$$

Note that  $M$  is completely arbitrary so that (3) only governs the rate at which  $\Delta t \rightarrow 0$  and not the size of  $\Delta t$ . In particular, restriction (3) is not a typical artifact of the discrete Gronwall inequality since it does not depend problem data. The error estimate is obtained by a bootstrap argument that utilizes the error in the energy norm. Although such estimates have been proved for BECE in [32], the analysis of CNLE is distinctly different because CN methods only give *a priori* control of *average* velocities  $\mathbf{u}^{n+1/2}$  rather than pointwise velocities  $\mathbf{u}^{n+1}$  (e.g. BE methods) in  $l^2(H^1)$ . Our analysis depends on

- Estimate (75) - bound test-function of convective term in  $L^2$  (Section 1.1.1)
- CN *a priori* estimates - introduce  $\Delta t$ -restriction (3) (Section 1.1.2)
- Stokes projection - preserve optimal convergence rate (Section 1.1.3).

Indeed, we derive and apply estimate (75)(b) of the *explicitly* skew-symmetric convective term; we obtain intermediate estimates in the convergence analysis of CNLE with limited options corresponding to limited control of *average* velocities (characteristic of CN methods) in  $l^2(H^1)$ ; and we exploit the Stokes projection to preserve the optimal convergence rate for the FE and CN discretization.

It is a source of current research to determine whether (3) is strictly necessary and to develop linear, implicit variants of usual CNLE (2)(a) that are guaranteed to avoid (3). Indeed, the extrapolated convecting velocity in (2)(a) is the source of  $\Delta t$ -restriction for convergence in  $l^\infty(H^1)$ . Moreover, higher-order extrapolations reduce the modeling error of extrapolation. Consequently, we investigate the arbitrary-order extrapolation herein: for some  $n_0 \geq 0$  and  $a_i \geq 0$  for  $0 \leq i \leq n_0$ , consider

$$\int \frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} \cdot \nabla \frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} \cdot \mathbf{v} \approx \int (a_0 \mathbf{u}^n + \dots + a_{n_0} \mathbf{u}^{n-n_0}) \cdot \nabla \frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2} \cdot \mathbf{v}.$$

We consider the important case when the nonlocal compatibility condition is not satisfied (addressed by Heywood and Rannacher in [19, 20] and more recently, e.g., by He in [12, 14] and He and Li in [15, 16]). Suppose, for example, that  $p^0$  be the solution of (well-posed) Neumann problem

$$\begin{cases} \Delta p^0 = \nabla \cdot (\mathbf{f}^0 - \mathbf{u}^0 \cdot \nabla \mathbf{u}^0), & \text{in } \Omega, \\ \nabla p^0 \cdot \hat{\mathbf{n}}|_{\partial\Omega} = (\Delta \mathbf{u}^0 + \mathbf{f}^0 - \mathbf{u}^0 \cdot \nabla \mathbf{u}^0) \cdot \hat{\mathbf{n}}|_{\partial\Omega}. \end{cases} \quad (4)$$

In order to avoid the accompanying factor  $\min \{t^{-1}, 1\}$  in the estimates presented in Section 3, the following compatibility condition is necessarily required (e.g. see [19], Corollary 2.1):

$$\nabla p^0|_{\partial\Omega} = (\Delta \mathbf{u}^0 + \mathbf{f}^0 - \mathbf{u}^0 \cdot \nabla \mathbf{u}^0)|_{\partial\Omega}. \quad (5)$$

Replacing (4) with (4)(a), (5) defines an overdetermined Neumann-type problem. Condition (5) is a nonlocal condition relating  $\mathbf{u}^0$  and  $\mathbf{f}^0$ . Condition (5) is satisfied for several practical applications including startup from rest with zero force,  $\mathbf{u}^0 = 0$ ,  $\mathbf{f}^0 = 0$ . In general, however, condition (5) cannot be verified. In this case, it is shown that, e.g.,  $\|\mathbf{u}_t(\cdot, t)\|_1, \|\mathbf{u}(\cdot, t)\|_3 \rightarrow \infty$  as  $t \rightarrow 0^+$ .

In Section 1.1, we provide an overview of CN time-stepping schemes for NSE approximation and explain our methodology for CNLE convergence analysis and corresponding improved estimates. We provide the mathematical setting for CNLE in Section 2 for both the continuous and discrete function spaces (Section 2.1). In Section 2.2, we compile the fundamental estimates and assumptions of the FE space and extrapolated convecting velocity. Section 3 contains the main results of our report. Section 4 contains the proofs of these results. In Section 3.2, we compile the fundamental approximations and identities required for our error analysis. Section 3.3 is devoted to analysis of the trilinear convective term and the explicitly skew-symmetric convective term used in CNLE. In Section 3.4, we present the elliptic and Stokes projections.

**1.1. Remark on improved estimate.** The key difference between our convergence proof for CNLE and that of CNFE is the resulting intermediate estimate: for approximations  $\mathbf{U}_h^n$  and constants  $\kappa^n > 0$ ,

$$\text{CNFE} \Rightarrow \|\mathbf{U}_h^N\|^2 + \dots \leq \sum_{n=0}^N \kappa^n \|\mathbf{U}_h^n\|^2 + \dots \quad (6)$$

$$\text{CNLE} \Rightarrow \|\mathbf{U}_h^N\|^2 + \dots \leq \sum_{n=0}^{N-1} \kappa^n \|\mathbf{U}_h^n\|^2 + \dots \quad (7)$$

Notice that the term  $\|\mathbf{U}_h^N\|^2$  is included in the right-hand-side of (6), but not of (7). Estimates of the form (6) require a discrete Gronwall inequality (Lemma 3.14) to proceed, which is the source of a  $\Delta t$ -restriction. Conversely, estimates of the form

(7) allow application of an alternate discrete Gronwall inequality (Lemma 3.15), which does not require a  $\Delta t$ -restriction.

**1.1.1. Key estimate for CNLE error analysis.** The key difficulty resolved in our CNLE proof (resulting in suboptimal results reported in previous error analyses) is associated with the extrapolated convecting velocity  $\frac{3}{2}\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1}$ . Once again consider the error equation for CNLE:

$$\begin{aligned} \|\mathbf{U}_h^N\|^2 + Re^{-1}\Delta t \sum_n \|\nabla \frac{\mathbf{U}_h^{n+1} + \mathbf{U}_h^n}{2}\|^2 + \dots \\ = \Delta t \sum_n \int (\frac{3}{2}\mathbf{U}_h^n - \frac{1}{2}\mathbf{U}_h^{n-1}) \cdot \nabla \mathbf{u} \cdot \frac{\mathbf{U}_h^{n+1} + \mathbf{U}_h^n}{2} + \dots \end{aligned} \quad (8)$$

To derive an *a priori* estimate from (8), each  $\mathbf{U}_h^n$  from the right-hand side must be absorbed into the left-hand side. However,  $\frac{3}{2}\mathbf{U}_h^n - \frac{1}{2}\mathbf{U}_h^{n-1}$  cannot be written as a sum of averages. Indeed, suppose that  $\mathbf{U}_h^n = -\mathbf{U}_h^{n+1} \neq 0$  so that  $\|\nabla \frac{\mathbf{U}_h^{n+1} + \mathbf{U}_h^n}{2}\| = 0$  whereas  $\|\nabla \mathbf{U}_h^{n+1}\| > 0$ . Then, in this case, it is impossible to absorb any factor of  $\sum_n \|\nabla (\frac{3}{2}\mathbf{U}_h^n - \frac{1}{2}\mathbf{U}_h^{n-1})\|^2 > 0$  into  $\sum_n \|\nabla \frac{\mathbf{U}_h^{n+1} + \mathbf{U}_h^n}{2}\|^2 = 0$ . To proceed from (8) requires care so that only  $L^2$ -norms of  $\frac{3}{2}\mathbf{U}_h^n - \frac{1}{2}\mathbf{U}_h^{n-1}$  are introduced when majorizing the right-hand side. Indeed  $\sum_n \kappa_n \|\mathbf{U}_h^n\|^2$  can be *absorbed* via the discrete Gronwall Lemma. Assuming that  $\mathbf{u} \in H^2$ , the key estimate to prove is

$$|c_h(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C \|\mathbf{u}\|_{H^r} \|\mathbf{v}\|_{H^2} \|\mathbf{w}\|_{H^{1-r}} \quad \forall \mathbf{u} \in H^r, \mathbf{v} \in H^2, \mathbf{w} \in H^{1-r}$$

for some  $C > 0$  and  $r = 0$  and  $1$  where  $c_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v})$  is the explicitly skew-symmetric convective term (see estimates (74)(b), (75)(b)).

**1.1.2. Introduction of  $\Delta t$ -restriction (3).** It is illuminating to introduce the BE scheme to highlight the difficulty in convergence estimates for CN schemes in higher order norms. Let  $i = 1$  for BE and  $i = 2$  for CN. Write  $z^{n+1/2} = \frac{1}{2}(z^{n+1} + z^n)$  and

$$\int \mathbf{u}^{n+1/i} \cdot \nabla \mathbf{u}^{n+1/i} \cdot \mathbf{v} \approx \int \xi^n(\mathbf{u}) \cdot \nabla \mathbf{u}^{n+1/i} \cdot \mathbf{v}, \quad \xi^n(\mathbf{u}) := a_0 \mathbf{u}^n + \dots + a_{n_0} \mathbf{u}^{n-n_0}. \quad (9)$$

Note that  $\xi^n(\mathbf{u}) = \mathbf{u}^n$  and  $\xi^n(\mathbf{u}) = \frac{3}{2}\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1}$  in (9) for BECE and CNLE respectively. The *energy difference* due to the numerical extrapolation (9) is the source of the  $\Delta t$ -restriction (3) for CNLE. Indeed, BECE velocities are shown to converge unconditionally in  $l^\infty(H^1)$  (see e.g. [32]). Let  $\mathbf{e}$  represent the fully discrete velocity error for CNLE or BECE. We show herein for CNLE (and it is known for BECE) that

$$\max_n \|\mathbf{e}^n\| + (Re^{-1}\Delta t \sum_n \|\nabla \mathbf{e}^{n+1/i}\|^2)^{1/2} \leq C(h^k + \Delta t^i) \quad (10)$$

for some  $C > 0$  without any  $\Delta t$ -restriction. The key difference between our convergence proof for CNLE and that for BECE is the resulting error equation: for

approximations  $\mathbf{U}_h^n$ ,

$$\begin{aligned}
 & Re^{-1} \|\nabla \mathbf{U}_h^N\|^2 + \Delta t \sum_n \left\| \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t} \right\|^2 + \dots \\
 &= \begin{cases} \Delta t \sum_n \int \mathbf{e}^n \cdot \nabla \mathbf{e}^{n+1} \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t} + \dots & \text{BECE} \\ \Delta t \sum_n \int \left( \frac{3}{2} \mathbf{e}^n - \frac{1}{2} \mathbf{e}^{n-1} \right) \cdot \nabla \frac{\mathbf{e}^{n+1} + \mathbf{e}^n}{2} \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t} + \dots & \text{CNLE} \end{cases} \quad (11)
 \end{aligned}$$

Estimate (10) gives us *a priori* control in  $l^2(H^1)$  of two terms  $\mathbf{e}^n, \mathbf{e}^{n+1}$  in (11)(a). However, we only have *a priori* control of  $\frac{1}{2}(\mathbf{e}^{n+1} + \mathbf{e}^n)$  in (11)(b) since  $\frac{3}{2}\mathbf{e}^n - \frac{1}{2}\mathbf{e}^{n-1}$  cannot be written as a sum of averages. Indeed, suppose that  $\mathbf{e}^n = -\mathbf{e}^{n+1} \neq 0$ . Then  $\|\nabla \frac{\mathbf{e}^{n+1} + \mathbf{e}^n}{2}\| = 0$  whereas  $\|\nabla \mathbf{e}^{n+1}\| > 0$ . In this case it is impossible to bound  $\sum_n \|\nabla(\frac{3}{2}\mathbf{e}^n - \frac{1}{2}\mathbf{e}^{n-1})\|^2 > 0$  above by any factor of  $\sum_n \|\nabla \frac{\mathbf{e}^{n+1} + \mathbf{e}^n}{2}\|^2 = 0$ . The remaining term in (11)(a) and 2 terms in (11)(b) must be absorbed to the left-hand side of the estimate or with the Gronwall Lemma. The limited control of the CNLE term  $\int (\frac{3}{2}\mathbf{e}^n - \frac{1}{2}\mathbf{e}^{n-1}) \cdot \nabla \frac{\mathbf{e}^{n+1} + \mathbf{e}^n}{2} \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}$  leads to the restriction (3).

**1.1.3. Preserving optimal convergence rate in  $l^\infty(H^1)$ .** We utilize the Stokes projections (79) in the convergence analysis of Theorems 3.8, 3.10. The Stokes projection requires additional regularity of the pressure  $p$ , but is necessary to establish the optimal convergence rate for velocity in  $l^\infty(H^1)$  reported in Theorem 3.10. The crucial estimate involves the error in the pressure: for each  $n \geq 0$ , fix  $\tilde{q}_h^{n+1} \approx p^{n+1}$  and  $\mathbf{U}_h^{n+1} \in H_0^1$  so that

$$(p^{n+1} - \tilde{q}_h^{n+1}, \nabla \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) \leq \begin{cases} \|p^{n+1} - \tilde{q}_h^{n+1}\| \|\nabla \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}\| \\ \|p^{n+1} - \tilde{q}_h^{n+1}\|_1 \|\frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}\| \end{cases} \quad (12)$$

The first option in (12) must be avoided, because we have no *a priori* control of  $\|\nabla \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}\|$ . The second option (12) is applicable, but ultimately leads to a suboptimal error estimate. Indeed, approximation theory for FE functions suggests

$$\|p^{n+1} - \tilde{q}_h^{n+1}\|_m \leq Ch^{s+1-m}, \quad s = \text{degree of FE space} \quad (13)$$

for some  $C > 0$  so that a factor of  $h$  is *lost* in the case  $m = 1$ . Alternatively, let  $(\tilde{\mathbf{v}}_h^{n+1}, \tilde{q}_h^{n+1}) \approx (\mathbf{u}^{n+1}, p^{n+1})$  be the Stokes projection. Then

$$Re^{-1} (\nabla(\mathbf{u}^{n+1} - \tilde{\mathbf{v}}_h^{n+1}), \nabla \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) - (p^{n+1} - \tilde{q}_h^{n+1}, \nabla \cdot \frac{\mathbf{U}_h^{n+1} - \mathbf{U}_h^n}{\Delta t}) = 0. \quad (14)$$

Identity (14) eliminates the need to bound (12). Instead, the error is shifted to the time derivative of the Stokes projection and requires

$$\left\| \frac{(p^{n+1} - \tilde{q}_h^{n+1}) - (p^n - \tilde{q}_h^n)}{\Delta t} \right\| \leq Ch^{s+1} \quad (15)$$

for some  $C > 0$ .



**1.2. Overview of CN schemes.** There are many analyses of CN time-stepping methods for the NSE. Heywood and Rannacher [20] provide a well-cited and comprehensive analysis of CNFE. The 2nd and 3rd order CNLE methods are introduced and analyzed in [3, 4]. Numerous error analyses of CNLE have been provided since: e.g. multilevel based CNLE [18, 21], LES turbulence modeling [6], stochastic CNLE [7], and a stabilized CNLE method [24]. Each of these analyses requires, explicitly stated or implicitly, a  $\Delta t$ -restriction of the form (1) to guarantee convergence. For instance, in [6], the first Gronwall Lemma 3.14 must be applied with  $\Delta t$ -restriction (p. 223) since Equation (4.11) includes the pertinent term up to and including the current time-step (see 4th-term on RHS of Equation (4.11)). The convergence analysis in [12–17] requires  $\Delta t \leq c_T$  or  $\Delta t \leq c_T |\log h|^{-1}$ . Convergence estimates for fully-discrete BECE is derived and family of semi-discrete multi-step CNLE-type methods are derived in [32] and [10] respectively. Concurrent to our work, the authors of [26] correctly perform the convergence analysis in energy norm, but not in higher-orders and do not consider general case when the nonlocal compatibility condition (4), (5) is not satisfied.

Applications of CNLE are also widespread: e.g. stability analysis of NSE and MHD equations with CNLE in [2], a 1st order CNLE applied in conjunction with a coupled multigrid and pressure Schur complement scheme for the NSE is proposed in [23], a velocity-vorticity formulation of CNLE analyzed in [29], and the NS- $\alpha$ /NS- $\omega$  regularization method with CNLE [25]. The CN method is also applied, for example, to a general class of non-stationary partial differential equations encompassing reaction-diffusion type equations including the nonlinear Sobolev equations [28] and the Ginzburg-Landau model [22]. Time-step restrictions of type (1)(b) (where  $Re$  has a different meaning) are implicitly required in the convergence analyses of these discrete models.

A CN/Adams-Bashforth (CN-AB) time-stepping, scheme is another linear variant of CNFE. Unlike CNLE, CN-AB is explicit in the nonlinearity and only *conditionally* stable [17] (i.e. a  $\Delta t$ -restriction of form (1)(a) is required for *stability*). CN-AB is a popular method for approximating NS flows because it is fast and easy to implement. Each time-step requires only one discrete Stokes system and linear solve. For example, CN-AB is used to model turbulent flows induced by wind turbine motion [31], turbulent flows transporting particles [27], and reacting flows in complex geometries (e.g. gas turbine combustors) [1].

Lastly, the compatibility condition (4), (5) is not satisfied in general, see [20] for an overview of this problem. When the (4), (5) is not satisfied, we have limited regularity that greatly restricts convergence analysis of high-order methods (both in time and *space*), e.g.  $\partial_t^{(2)} \mathbf{u} \notin L^2(H^1)$ . Corresponding regularity in the case that (4), (5) is satisfied is assumed the analyses of [3, 4, 6, 7, 10, 24, 26, 32]. See [12, 14–16, 19, 20, 30] for a rigorous treatment of the general case when the compatibility condition is not satisfied. We also provide details herein.

## 2. Problem formulation

Let  $\mathbf{a} := (a_0, a_1, \dots, a_{n_0}) \in \mathbb{R}^{n_0+1}$  for some  $n_0 \in \{0\} \cup \mathbb{N}$  be equipped with the standard  $l^q$  norm denoted by  $|\mathbf{a}|_q$  for  $1 \leq q \leq \infty$ . Fix  $p \geq 1$ . Let  $L^p(\Omega)$  denote the linear space of all real Lebesgue-measurable functions  $\mathbf{u}$  and bounded in the usual norm denoted by  $\|\mathbf{u}\|_{L^p(\Omega)}$ . Let  $(\cdot, \cdot)_\Omega$  and  $\|\cdot\|_\Omega$  be the standard  $L^2(\Omega)$ -inner product and norm. Fix  $k \in \mathbb{R}$ . The Sobolev space  $W^{k,p}(\Omega)$  is equipped



with the usual norm denoted by  $\|\mathbf{u}\|_{W^{k,p}(\Omega)}$ . Identify  $\|\cdot\|_{k,p,\Omega} := \|\cdot\|_{W^{k,p}(\Omega)}$ ,  $H^k(\Omega) := W^{k,2}(\Omega)$ ,  $\|\cdot\|_{k,\Omega} := \|\cdot\|_{W^{k,2}(\Omega)}$  with  $\|\cdot\|_{k,\Omega}$  the corresponding semi-norm. Let the context determine whether  $W^{k,p}(\Omega)$  denotes a scalar, vector, or tensor function space. For example let  $\mathbf{v} : \Omega \rightarrow \mathbb{R}^d$ . Then,  $\mathbf{v} \in H^1(\Omega)$  implies that  $\mathbf{v} \in H^1(\Omega)^d$  and  $\nabla \mathbf{v} \in H^1(\Omega)$  implies that  $\nabla \mathbf{v} \in H^1(\Omega)^{d \times d}$ . Define

$$H_0^1(\Omega) := \{\mathbf{v} \in H^1(\Omega) : \mathbf{v}|_{\partial\Omega} = 0\}, \quad V(\Omega) := \{\mathbf{v} \in H_0^1(\Omega) : \nabla \cdot \mathbf{v} = 0\}.$$

The dual space of  $H_0^1(\Omega)$  is denoted  $W^{-1,2}(\Omega) := (H_0^1(\Omega))'$  and equipped with the norm

$$\|\mathbf{f}\|_{-1,\Omega} := \sup_{0 \neq \mathbf{v} \in H_0^1(\Omega)} \frac{\langle \mathbf{f}, \mathbf{v} \rangle_{W^{-1,2}(\Omega) \times H_0^1(\Omega)}}{|\mathbf{v}|_{1,\Omega}}.$$

Define

$$L_0^2(\Omega) := \{q \in L^2(\Omega) : (q, 1)_\Omega = 0\}.$$

For brevity, omit  $\Omega$  in the definitions above. For example,  $(\cdot, \cdot) = (\cdot, \cdot)_\Omega$ ,  $H^1 = H^1(\Omega)$ , and  $V = V(\Omega)$ .

Fix time  $T > 0$  and  $m \geq 1$ . Let  $W^{m,q}(0, T; W^{k,p}(\Omega))$  denote the linear space of all Lebesgue measurable functions from  $(0, T)$  onto  $W^{k,p}$  equipped with and bounded in the norm

$$\|\mathbf{u}\|_{W^{m,q}(0,T;W^{k,p})} := \left( \int_0^T \sum_{i=0}^m \|\partial_t^{(i)} \mathbf{u}(\cdot, t)\|_{W^{k,p}}^q dt \right)^{1/q}.$$

Write  $W^{m,q}(W^{k,p}) = W^{m,q}(0, T; W^{k,p}(\Omega))$  and  $C^m(W^{k,p}) = C^m([0, T]; W^{k,p}(\Omega))$ .

**2.1. Discrete function setting.** Fix  $h > 0$ . Let  $\mathcal{T}_h$  be a family of subdivisions (e.g. triangulation) of  $\bar{\Omega} \subset \mathbb{R}^d$  satisfying  $\bar{\Omega} = \bigcup_{E \in \mathcal{T}_h} E$  so that  $\text{diameter}(E) \leq h$  and any two closed elements  $E_1, E_2 \in \mathcal{T}_h$  are either disjoint or share exactly one face, side, or vertex. Suppose further that  $\mathcal{T}_h$  is quasi-uniformly regular as  $h \rightarrow 0$ . See [5] (Definition 4.4.13) for a precise definition and treatment of the inherited properties of such a space (see Appendix II.A in [11] for more on this subject in context of Stokes problem). For example,  $\mathcal{T}_h$  consists of triangles for  $d = 2$  or tetrahedra for  $d = 3$  that are nondegenerate as  $h \rightarrow 0$ .

Let  $X_{h,\cdot} \subset (H^1)^d$  and  $Q_{h,\cdot} \subset L^2$  be a FE space. For example, let  $X_{h,\cdot}$  and  $Q_{h,\cdot}$  be continuous, piecewise (on each  $E \in \mathcal{T}_h$ ) polynomial spaces. Define  $X_h := X_{h,\cdot} \cap H_0^1$ ,  $Q_h := Q_{h,\cdot} \cap L_0^2$ . The discretely divergence-free space is given by

$$V_h = \{\mathbf{v}_h \in X_h : (q_h, \nabla \cdot \mathbf{v}_h) = 0 \quad \forall q_h \in Q_{h,\cdot}\}.$$

Note that in general  $V_h \not\subset V$  (e.g. Taylor-Hood elements).

Set  $0 = t^0 < t^1 < \dots < t^n = T < \infty$  with  $\Delta t = t^n - t^{n-1}$ . Write  $z^n := z(t^n)$  and  $z^{n+1/2} := \frac{1}{2}(z(t^{n+1}) + z(t^n))$ . Define

$$\|\mathbf{u}\|_{l^q(m_1, m_2; W^{k,p})} := \begin{cases} (\Delta t \sum_{n=m_1}^{m_2} \|\mathbf{u}^n\|_{k,p}^q)^{1/q}, & q \in [1, \infty) \\ \max_{m_1 \leq n \leq m_2} \|\mathbf{u}^n\|_{k,p}, & q = \infty \end{cases}$$

for any  $0 \leq n = m_1, m_1 + 1, \dots, m_2 \leq N$ . Write  $\|\mathbf{u}\|_{l^q(W^{k,p})} = \|\mathbf{u}\|_{l^q(0, N; W^{k,p})}$ . We say that  $\mathbf{u} \in l^q(m_1, m_2; W^{k,p})$  if the associated norm defined above stays finite as  $\Delta t \rightarrow 0$ . Define the discrete time-derivative

$$\partial_{\Delta t}^{n+1} \mathbf{v} := \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t}.$$

Explicit skew-symmetrization of the convective term in NS-type equations ensures stability of the corresponding numerical approximation:

$$c_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) := \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) - \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}). \quad (16)$$

Fix  $a_i \in \mathbb{R}$  for  $i = -1, 0, 1, \dots, n_0 \geq -1$  and  $n \in \{0\} \cup \mathbb{N}$ . Define  $\xi^n(\mathbf{u})$  so that

$$c_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) \approx c_h(\xi^n(\mathbf{u}), \mathbf{v}, \mathbf{w}), \quad \xi^n(\mathbf{u}) := a_{-1}\mathbf{u}^{n+1} + a_0\mathbf{u}^n + \dots + a_{n_0}\mathbf{u}^{n-n_0}.$$

To summarize,

$$\begin{aligned} \text{No linearization} &\Rightarrow a_{-1} = 1, \quad a_i = 0 \text{ for all } i \geq 0 \\ \text{Linearization} &\Rightarrow a_{-1} = 0, \quad a_i \neq 0 \text{ for some } i \geq 0 \end{aligned}$$

For example,

$$\begin{aligned} \xi^n(\mathbf{u}) &= \mathbf{u}^n &\Rightarrow \xi^n(\mathbf{u}) &= \mathbf{u}(\cdot, t^{n+1/2}) + \mathcal{O}(\Delta t) \\ \xi^n(\mathbf{u}) &= \frac{1}{2}(3\mathbf{u}^n - \mathbf{u}^{n-1}) &\Rightarrow \xi^n(\mathbf{u}) &= \mathbf{u}(\cdot, t^{n+1/2}) + \mathcal{O}(\Delta t^2) \\ \xi^n(\mathbf{u}) &= \frac{1}{8}(15\mathbf{u}^n - 10\mathbf{u}^{n-1} + 3\mathbf{u}^{n-2}) &\Rightarrow \xi^n(\mathbf{u}) &= \mathbf{u}(\cdot, t^{n+1/2}) + \mathcal{O}(\Delta t^3) \end{aligned}$$

CNLE is a particularly attractive method because it is  $\Delta t^2$ -accurate, implicit in the nonlinearity (a source of stiffness), and linearized which avoids issues of nonlinear solvers converging and greatly reduces the computational cost. Fix the kinematical viscosity  $\nu > 0$ . Note that  $\nu \propto Re^{-1}$ .

**Problem 2.1** (CNLE). Suppose that  $\mathbf{u}_h^i \in V_h$  for  $i = 0, 1, \dots, n_0$  and  $p_h^{n_0} \in Q_h$ . For each  $n = n_0, n_0 + 1, \dots, N - 1$ , find  $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in X_h \times Q_h$  satisfying

$$\begin{aligned} & \left( \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h \right) + c_h(\xi^n(\mathbf{u}_h), \mathbf{u}_h^{n+1/2}, \mathbf{v}_h) \\ & + \nu(\nabla \mathbf{u}_h^{n+1/2}, \nabla \mathbf{v}_h) - (p_h^{n+1/2}, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}^{n+1/2}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in X_h \end{aligned} \quad (17)$$

$$(q_h, \nabla \cdot \mathbf{u}_h^{n+1}) = 0, \quad \forall q_h \in Q_h. \quad (18)$$

**Remark 2.2.** Note that  $\xi^n(\mathbf{u}_h) = \mathbf{u}_h^{n+1/2}$  defines the CNFE method analyzed in e.g. [20],  $\xi^n(\mathbf{u}_h) = \frac{1}{2}(3\mathbf{u}_h^n - \mathbf{u}_h^{n-1})$  defines the CNLE method of e.g. [3, 13, 24], and  $\xi^n(\mathbf{u}_h) = \mathbf{u}_h^{n-1/2}$  the  $\mathcal{O}(\Delta t)$  CNLE method of e.g. [6].

**2.2. Assumptions and approximations.** Let  $C > 0$  be a generic data-independent constant throughout; specifically,  $C$  is independent of  $h, \Delta t, \nu \rightarrow 0$ . Error estimates for the elliptic projection (78) and Stokes projection (80) in  $L^2$  and  $W^{-1,2}$  require regularity of solutions to the following auxiliary problem.

**Assumption 2.3.** Given  $\theta \in W^{-1,2}$ , find  $(\mathbf{w}_\theta, r_\theta) \in H_0^1 \times L_0^2$  satisfying

$$(\nabla \mathbf{w}_\theta, \nabla \mathbf{v}) - (r_\theta, \nabla \cdot \mathbf{v}) - (q, \nabla \cdot \mathbf{w}) = (\theta, \mathbf{v}), \quad \forall (\mathbf{v}, q) \in H_0^1 \times L^2.$$

This problem is well-known to be well-posed. Suppose further that  $(\mathbf{w}_\theta, r_\theta) \in (H^{m+2} \cap V) \times (H^{m+1} \cap L_0^2)$  satisfy

$$\|\mathbf{w}_\theta\|_{m+2} + \|r_\theta\|_{m+1} \leq C\|\theta\|_m \quad (19)$$

when  $m = 0, 1$  and  $\theta \in H_0^m$  (with  $H_0^0 = L^2$ ).

Indeed, (19) is true if  $\Omega$  is smooth enough.

Preserving an abstract framework for the FE spaces, we assume that  $X_h \times Q_h$  inherit several fundamental approximation properties.

**Assumption 2.4.** The FE spaces  $X_h \times Q_h$  satisfy:

**Uniform inf-sup (LBB) condition:**

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in X_h} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{|\mathbf{v}_h|_1 \|q\|} \geq C > 0 \quad (20)$$

**FE-approximation:**

$$\begin{aligned} \inf_{\mathbf{v}_h \in X_h} \|\mathbf{u} - \mathbf{v}_h\|_1 &\leq Ch^k \|\mathbf{u}\|_{k+1} \\ \inf_{q_h \in Q_h} \|p - q_h\| &\leq Ch^{s+1} \|p\|_{s+1} \end{aligned} \quad (21)$$

for  $k \geq 0$ , and  $s \geq -1$  when  $\mathbf{u} \in H^{k+1} \cap H_0^1$ ,  $p \in H^{s+1} \cap L_0^2$

**Inverse-estimate:**

$$|\mathbf{v}_h|_1 \leq Ch^{-1} \|\mathbf{v}_h\|, \quad \forall \mathbf{v}_h \in X_h \quad (22)$$

The well-known Taylor-Hood mixed FE is one such example satisfying Assumption 2.4. Estimates in (23), (24), (25), (28) stated below are used in proving error estimates for time-dependent problems. First define

$$\sigma(t) := \min\{1, t\}.$$

Then for any  $n = n_0, n_0 + 1, \dots, N - 1$ ,  $k \geq -1$ ,

$$\|\partial_{\Delta t}^{n+1} \mathbf{u}\|_k^2 \leq \Delta t^{-1} \int_{t^n}^{t^{n+1}} \|\partial_t \mathbf{u}(\cdot, t)\|_k^2 dt \quad (23)$$

$$\|\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})\|_k^2 \leq C \Delta t^3 \sigma(t^{n+1/2})^{-2} \int_{t^n}^{t^{n+1}} \sigma(t)^2 \|\partial_t^{(2)} \mathbf{u}(\cdot, t)\|_k^2 dt \quad (24)$$

$$\|\partial_{\Delta t}^{n+1} \mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}\|_k^2 \leq C \Delta t^3 \sigma(t^{n+1/2})^{-2} \int_{t^n}^{t^{n+1}} \sigma(t)^2 \|\partial_t^{(3)} \mathbf{u}(\cdot, t)\|_k^2 dt \quad (25)$$

when  $\partial_t \mathbf{u} \in L^2(H^k)$ ,  $t^2 \partial_t^{(2)} \mathbf{u}(\cdot, t) \in L^2(H^k)$ ,  $t^2 \partial_t^{(3)} \mathbf{u}(\cdot, t) \in L^2(H^k)$ . Derivation of these estimates follows from application of an appropriate Taylor expansion with integral remainder, see Appendix B. Moreover, (24) and (25) are replaced by

$$\|\mathbf{u}^{n+1/2} - \mathbf{u}(\cdot, t^{n+1/2})\|_k^2 \leq C \Delta t^3 \int_{t^n}^{t^{n+1}} \|\partial_t^{(2)} \mathbf{u}(\cdot, t)\|_k^2 dt \quad (26)$$

$$\|\partial_{\Delta t}^{n+1} \mathbf{u} - (\partial_t \mathbf{u})^{n+1/2}\|_k^2 \leq C \Delta t^3 \int_{t^n}^{t^{n+1}} \|\partial_t^{(3)} \mathbf{u}(\cdot, t)\|_k^2 dt \quad (27)$$

if  $\partial_t^{(2)} \mathbf{u} \in L^2(H^k)$ ,  $\partial_t^{(3)} \mathbf{u} \in L^2(H^k)$  respectively. The operator  $\xi^n(\mathbf{u})$  should be chosen to preserve the  $\Delta t^2$ -convergence rate of the fully-nonlinear CN scheme. This is made precise by assuming (28) holds.

**Assumption 2.5.** Suppose that  $t^2 \partial_t^{(2)} \mathbf{u} \in L^2(H^k)$  for some  $k \geq 0$ . Then for each  $n = n_0, n_0 + 1, \dots, N - 1$ ,

$$\|\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})\|_k^2 \leq C \Delta t^3 \sigma(t^{n+1/2})^{-2} \int_{t^{n-n_0}}^{t^{n+1}} \sigma(t)^2 \|\mathbf{u}(\cdot, t)\|_k^2 dt \quad (28)$$

Note that if  $\partial_t^{(2)} \mathbf{u} \in L^2(H^k)$ , then (28) becomes

$$\|\xi^n(\mathbf{u}) - \mathbf{u}(\cdot, t^{n+1/2})\|_k^2 \leq C \Delta t^3 \int_{t^{n-n_0}}^{t^{n+1}} \|\mathbf{u}(\cdot, t)\|_k^2 dt \quad (29)$$

We summarize the continuous-in-space and discrete-in-time counterparts to the regularity results of Theorems 2.2, 2.3., 2.4 reported in [19] when the nonlocal compatibility condition (4), (5) is *not* satisfied. The same results (with extensions) are also reported and applied in [12, 14–16]). We summarize the results here as an assumption. The results follow if  $\mathbf{u} \in L^\infty(H^1)$  and  $\partial\Omega$  and problem data are smooth enough.

**Assumption 2.6.** *Suppose that for any  $i \geq 0$ ,  $k \geq -1$ , and  $2i + k > 0$  that*

$$\int_{t^{n_0}}^T \sigma(t)^{2i+k-3} \|\partial_t^{(i)} \mathbf{u}(\cdot, t)\|_k^2 dt < \infty$$

and

$$\begin{aligned} & \Delta t \sum_{n=1}^N \sigma(t^n)^{2i+k-2} \|\partial_t^{(i)} \mathbf{u}^n\|_{k+1}^2 \\ & + \max_{1 \leq n \leq N} \sigma(t^n)^{2i+k-2} \|\mathbf{u}^n\|_k^2 + \Delta t \sum_{k=1}^N \sigma(t^n)^{2i+k-2} \|\partial_t^{(i)} p^n\|_k^2 < \infty. \end{aligned}$$

### 3. Convergence estimate for CNLE

We state the main results in this section. A list of the constants found in these theorems (referenced throughout this section) are compiled in Section 3.1. The error equation required for the convergence proofs is derived in Section 4. Throughout, require that the continuous problem data minimally satisfies  $\mathbf{u}^0 \in H^2 \cap V$  and  $\mathbf{f} \in W^{1,\infty}(L^2)$ . In this setting, we state the NSE: find  $\mathbf{u} \in L^\infty(L^2) \cap L^2(H_0^1)$  and  $p \in W^{-1,\infty}(L_0^2)$  satisfying

$$\frac{d}{dt}(\mathbf{u}, \mathbf{v}) + (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in H_0^1 \quad (30)$$

$$\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0 \quad \text{a.e. } (\mathbf{x}, t) \in \Omega \times (0, T] \quad (31)$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \Omega. \quad (32)$$

See e.g. [9] and references therein for NSE existence and regularity results. Define  $\mathbf{e} := \mathbf{u} - \mathbf{u}_h$ . Proposition 3.1 provides sufficient conditions to conclude that  $\mathbf{e} \in l^\infty(L^2) \cap l^2(H^1)$ . Indeed, Proposition 3.1 holds for  $\mathbf{u} \in C^0([t^{n_0}, T]; H^1)$  without a Gronwall exponential factor (use *a priori* estimate derivable for  $\mathbf{u}_h^{n+1}$ ). Alternatively, the proof of Proposition 3.1 herein is obtained as an intermediate step in the proof of Theorem 3.5. We gain insight to the basic regularity requirements in our method of proof for the error estimate (34).

**Proposition 3.1.** *Suppose that the FE space satisfies Assumption 2.4. Suppose further that Assumption 2.3 is satisfied along with*

$$\mathbf{u} \in l^2(H^2) \cap l^\infty(V) \quad \partial_t \mathbf{u} \in l^2(n_0, N; W^{-1,2}), \quad p \in l^2(n_0, N; L_0^2),$$

and  $\partial_t \mathbf{u} \in L^2(t^{n_0}, T; H^1)$ . Then

$$\begin{aligned} & \|\mathbf{e}\|_{l^\infty(n_0+1, N; L^2)} + \nu^{1/2} \left( \Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2 \right)^{1/2} \leq C(\|\mathbf{u}^N\| + \nu^{1/2} \|\nabla \mathbf{u}\|_{l^2(n_0, N; L^2)}) \\ & + \exp(C\nu^{-1} \|\mathbf{u}\|_{l^2(n_0, N; H^2)}^2) (K_0 \|\mathbf{e}\|_{l^\infty(0, n_0; L^2)} + K_1 + K_2) \end{aligned} \quad (33)$$

The constants  $K_0, K_1, K_2 > 0$  are given in (41), (42), (43) respectively so that the right hand side of (33) remains bounded as  $h, \Delta t \rightarrow 0$ .

*Proof.* See Section 4.1. □

**Remark 3.2.** The regularity  $\partial_t \mathbf{u} \in L^2(t^{n_0}, T; H^1)$  assumption in Proposition 3.1 is more than necessary. Indeed, we can replace the elliptic projection with the  $L^2$ -projection to eliminate the time-derivative term  $(\partial_{\Delta t}^{n+1} \eta, \mathbf{U}_h^{n+1/2})$  (the source of the resulting regularity restriction) in the proof of Proposition 3.1. However, incorporation of the  $L^2$ -projection from  $V$  into  $V_h$  requires additional (technical) development and does not benefit the main error estimate in Theorem 3.5 for  $k > 2$ .

**Remark 3.3.** The exponential Gronwall factor  $\exp(C\nu^{-1} \|\mathbf{u}\|_{l^2(n_0, N; H^2)}^2)$  in  $K$  above typically takes the form  $\exp(C\nu^{-3} \|\nabla \mathbf{u}\|_{l^4(n_0, N; L^2)}^4)$ . However, a factor of  $\sum_n \|\nabla \mathbf{u}^n\|^2$  must be absorbed into  $\sum_n \|\nabla \mathbf{u}^{n+1/2}\|^2$  to proceed with the latter estimate. As discussed in Section 1.1.1, this is not possible in general for the usual CNLE (without a great restriction on time interval length  $T$ ). In particular, since the dissipative term for CN schemes has the form  $\nu \sum_n \|\nabla \mathbf{u}^{n+1/2}\|^2$  and the usual extrapolation for CNLE is  $\xi^n(\mathbf{u}) = \frac{3}{2}\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1}$  (which is not a sum of average velocities), errors manifested from the extrapolated velocity cannot be absorbed (as usual) into the dissipative term. Consequently all errors propagated from the extrapolated convecting velocity must be absorbed via the discrete Gronwall Lemma.

The optimal convergence rate proved in Theorem 3.5 requires that initial iterates  $\{\mathbf{u}_h^i\}_{i=0}^{n_0}$  are accurate enough. We make this precise in the following assumption.

**Assumption 3.4.** Fix  $k \geq 0$ ,  $s \geq -1$  and  $\alpha > 0$ . Suppose  $\{\mathbf{u}_h^i\}_{i=0}^{n_0}$  satisfy

$$K_0 \|\mathbf{e}\|_{l^\infty(0, n_0; L^2)} \leq \alpha(K_u h^k + K_p h^{s+1} + K_t \Delta t^2)$$

where  $K_0$ ,  $K_u$ ,  $K_p$ ,  $K_t > 0$  are given in (41), (44), (45), (46) respectively and remain bounded as  $h, \Delta t \rightarrow 0$ .

Note that Assumption 3.4 reduces to, when  $s = k - 1$ ,

$$\|\mathbf{e}\|_{l^\infty(0, n_0; L^2)} \leq \alpha(h^k + \Delta t^2).$$

The following theorem provides sufficient conditions to ensure CNLE converges optimally in the energy norm: e.g. under usual regularity conditions and  $s = k - 1$

$$\|\mathbf{e}\|_{l^\infty(n_0+1, N; L^2)} + \nu^{1/2} (\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2)^{1/2} \leq C(h^k + \Delta t^2)$$

without any  $\Delta t$ -restriction!

**Theorem 3.5.** Fix  $k > 0$ ,  $s > -1$ . Under the assumptions of Proposition 3.1, suppose further that Assumptions 2.5, 3.4 are satisfied along with  $\mathbf{u} \in l^\infty(H^k \cap H^2) \cap l^2(H^{k+1})$ ,  $\partial_t \mathbf{u} \in L^2(t^{n_0}, T; H^{k-1} \cap H^1) \cap l^\infty(n_0, N; H^1)$ ,  $\partial_t^{(2)} \mathbf{u} \in L^2(L^2)$ ,  $\partial_t^{(3)} \mathbf{u} \in L^2(t^{n_0}, T; W^{-1,2})$ , and  $p \in l^2(n_0, N; H^{s+1})$ . Then

$$\begin{aligned} \|\mathbf{e}\|_{l^\infty(n_0+1, N; L^2)} + \nu^{1/2} (\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2)^{1/2} &\leq G^N K_p h^{s+1} \\ &+ (C \|\mathbf{u}^N\|_k + C\nu^{1/2} \|\mathbf{u}\|_{l^2(n_0, N; H^{k+1})} + G^N K_u) h^k + G^N K_t \Delta t^2 \end{aligned} \quad (34)$$

where  $G^N := \exp(C\nu^{-1} \|\mathbf{u}\|_{l^2(n_0, N; H^2)}^2)$ . The constants  $K_u$ ,  $K_p$ ,  $K_t > 0$  are given in (44), (45), (46) respectively and remain bounded as  $h, \Delta t \rightarrow 0$ .

*Proof.* See Section 4.1. □

In the general case that the compatibility condition (4), (5) is not satisfied, the result of Theorem 3.5 holds under a reduced convergent rate: e.g. under usual regularity conditions and  $s = k - 1$

$$\|\mathbf{e}\|_{l^\infty(n_0+1, N; L^2)} + \nu^{1/2}(\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2)^{1/2} \leq C\sigma(t^1)^{-1}(\sigma(t^1)^{-(k-3)/2}h^k + \Delta t^2)$$

**Theorem 3.6.** Fix  $k > 0$ ,  $s > -1$ . Under the assumptions of Proposition 3.1, suppose further that Assumptions 2.5, 3.4, and 2.6 are satisfied. Then (34) holds with  $K_u$ ,  $K_p$ ,  $K_t$  replaced by  $\sigma(t^1)^{-(k-1)/2}\bar{K}_u$ ,  $\sigma(t^1)^{-(k-2)/2}\bar{K}_p$ , and  $\sigma(t^1)^{-1}\bar{K}_t$  respectively where the constants  $\bar{K}_u$ ,  $\bar{K}_p$ ,  $\bar{K}_t > 0$  are given in (47), (48), (49) respectively and remain bounded as  $h, \Delta t \rightarrow 0$ .

**Remark 3.7.** If we use the  $L^2$ -projection rather than elliptic projection in proving Theorem 3.6, the result is improvable so that  $K_u$  is replaced by  $\sigma(t^1)^{-(k-2)/2}\bar{K}_u$ .

*Proof.* See Section 4.1. □

An estimate for  $\Delta t \sum_n \|(\mathbf{e}^{n+1} - \mathbf{e}^n)/\Delta t\|^2$  is needed in the error analysis for pressure and the drag/lift forces by the fluid on embedded obstacles. Theorem 3.8 provides sufficient regularity of  $(\mathbf{u}, p)$  solving (81), (82), (83) to ensure  $\mathbf{u}_h \in l^\infty(H^1)$  and  $\partial_{\Delta t}\mathbf{u}_h \in l^2(L^2)$ . Note that the regularity  $\partial_t\mathbf{u} \in L^2(t^{n_0}, T; H^1)$ ,  $\partial_t p \in L^2(t^{n_0}, T; L^2)$  in Theorem 3.8 is a result of using the Stokes projection in corresponding proof herein and can be relaxed. The proof of Theorem 3.8 is obtained as an intermediate step in the proof of Theorem 3.10. Relaxing the regularity assumption (by using the  $L^2$ -projection or elliptic instead) leads to a suboptimal estimate. The additional regularity required here gives insight to the *extra* regularity required for the optimal error estimate (38).

**Theorem 3.8.** Under the assumptions of Proposition 3.1, suppose further that  $\mathbf{u} \in l^\infty(H^2)$ ,  $p \in l^\infty(H^1)$ ,  $\partial_t\mathbf{u} \in l^2(n_0, N; L^2) \cap L^2(t^{n_0}, T; H^1)$ , and  $\partial_t p \in L^2(t^{n_0}, T; L^2)$  so that

$$h^{-1}\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2 < \infty, \quad \text{as } h, \Delta t \rightarrow 0. \quad (35)$$

Then

$$\begin{aligned} & \|\partial_{\Delta t}\mathbf{e}\|_{l^2(n_0+1, N; L^2)} + \nu^{1/2}\|\nabla\mathbf{e}\|_{l^\infty(n_0+1, N; L^2)} \\ & \leq G^N(F_0\|\nabla\mathbf{e}\|_{l^\infty(0, n_0; L^2)} + F_1 + F_2) + C\nu^{-1/2}\|p^N\| \\ & \quad + C\nu^{1/2}\|\mathbf{u}^N\|_1 + Ch\nu^{-1}\|\partial_t p\|_{L^2(t^{n_0}, T; L^2)} + Ch\|\partial_t\mathbf{u}\|_{L^2(t^{n_0}, T; H^1)} \end{aligned} \quad (36)$$

where  $G^N := \exp(C\nu^{-1}\Delta t \sum_{n=n_0}^{N-1} (\|\mathbf{u}^{n+1/2}\|_2^2 + h^{-1}|\mathbf{e}^{n+1/2}|_1^2))$ . The constants  $F_0$ ,  $F_1$ ,  $F_2$  are given in (50), (51), (52) respectively so that the right hand side of (36) remains bounded as  $h, \Delta t \rightarrow 0$ .

*Proof.* See Section 4.2. □

The optimal convergence result proved in Theorem 3.10 requires that the initial iterates  $\{\mathbf{u}_h^i\}_{i=0}^{n_0}$  must be accurate enough. We make this precise in the following assumption.

**Assumption 3.9.** Fix  $k \geq 0$ ,  $s \geq -1$  and  $\alpha > 0$ . Suppose  $\{\mathbf{u}_h^i\}_i^{n_0}$  satisfy

$$F_0 \|\nabla \mathbf{e}\|_{l^\infty(0, n_0; L^2)} \leq \alpha(F_u h^k + F_p h^{s+1} + F_t \Delta t^2)$$

where  $F_0, F_u, F_p, F_t > 0$  are given in (50), (53), (54), (56) respectively.

Note that Assumption 3.9 reduces to, when  $s = k - 1$ ,

$$\|\nabla \mathbf{e}\|_{l^\infty(0, n_0; L^2)} \leq \alpha(h^k + \Delta t^2).$$

Therefore, under usual regularity conditions we show in Theorem 3.10 that

$$\|\partial_{\Delta t} \mathbf{e}\|_{l^2(n_0+1, N; L^2)} + \|\nabla \mathbf{e}\|_{l^\infty(n_0+1, N; L^2)} \leq C(h^k + \Delta t^2)$$

as long as  $\Delta t \leq Mh^{1/4}$  for any arbitrary  $M > 0$  (i.e. no  $\nu$ -dependence).

**Theorem 3.10.** Fix  $k > 0$ ,  $s > -1$ . Under the regularity of Theorem 3.8, suppose further that Assumptions 2.5, 3.9 are satisfied along with  $\mathbf{u} \in l^\infty(H^{k+1})$ ,  $\partial_t \mathbf{u} \in L^2(t^{n_0}, T; H^k \cap H^3) \cap l^\infty(n_0, N; L^2)$ ,  $\partial_t^{(2)} \mathbf{u} \in L^2(t^{n_0}, T; H^1)$ ,  $\partial_t^{(3)} \mathbf{u} \in L^2(t^{n_0}, T; L^2)$ ,  $p \in l^\infty(H^{s+1})$ ,  $\partial_t p \in L^2(t^{n_0}, T; H^s)$ , and

$$\Delta t \leq Mh^{1/4}, \quad \text{for any constant } M > 0 \text{ (no } \nu\text{-dependence).} \quad (37)$$

Then

$$\begin{aligned} & \|\partial_{\Delta t} \mathbf{e}\|_{l^2(n_0+1, N; L^2)} + \nu^{1/2} \|\nabla \mathbf{e}\|_{l^\infty(n_0+1, N; L^2)} \\ & \leq G^N (F_u h^k + F_p h^{s+1} + F_t \Delta t^2 + F_3 (\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2)^{1/2}) \end{aligned} \quad (38)$$

where  $G^N := \exp(C\nu^{-1} \Delta t \sum_{n=n_0}^{N-1} (\|\mathbf{u}^{n+1/2}\|_2^2 + h^{-1} |\mathbf{e}^{n+1/2}|_1^2))$ . The constants  $F_u, F_p, F_3, F_t > 0$  are given in (53), (54), (55), (56) respectively and remain finite as  $h, \Delta t \rightarrow 0$ .

*Proof.* See Section 4.2. □

In the general case that the compatibility condition (4), (5) is not satisfied, the result of Theorem 3.10 holds under a reduced convergent rate: e.g. under usual regularity conditions and  $s = k - 1$

$$\|\partial_{\Delta t} \mathbf{e}\|_{l^2(n_0+1, N; L^2)} + \nu^{1/2} \|\nabla \mathbf{e}\|_{l^\infty(n_0+1, N; L^2)} \leq C\sigma(t^1)^{-3/2} (\sigma(t^1)^{-(k-4)/2} h^k + \Delta t^2)$$

**Theorem 3.11.** Fix  $k > 0$ ,  $s > -1$ . Under the assumptions of Theorem 3.8, suppose further that Assumptions 2.5, 3.4, and 2.6 are satisfied. Then (38) holds with  $F_u, F_p, F_t$  replaced by  $\sigma(t^1)^{-(k-1)/2} \bar{F}_u$ ,  $\sigma(t^1)^{-(k-1)/2} \bar{F}_p$ , and  $\sigma(t^1)^{-3/2} \bar{F}_t$  respectively where the constants  $\bar{F}_u, \bar{F}_p, \bar{F}_t > 0$  are given in (57), (58), (59) respectively and remain bounded as  $h, \Delta t \rightarrow 0$ .

*Proof.* See Section 4.2. □

**Remark 3.12.** It is common to assume that  $\mathbf{u} \in L^\infty(W^{1,\infty})$  in the convergence analysis of NSE approximations (see e.g. [3, 24]). The conclusions of Proposition 3.1 3.1, Theorem 3.5, in addition to those of Theorems 3.8, 3.10 are preserved with the regularity condition  $\mathbf{u}(\cdot, t) \in H^2$  replaced by  $\mathbf{u}(\cdot, t) \in W^{1,\infty}$ . Regardless, the analysis of [24] suggests an  $h, \Delta t$ -restriction for optimal convergence in  $l^\infty(L^2) \cap l^2(H^1)$  (Theorem 3.1 in [24]) and a sub-optimal convergence estimate  $\|\mathbf{u} - \mathbf{u}_h\|_{l^\infty(H^1)} \leq \mathcal{O}(h^k + h^{s+1} + h^{-3/2} \Delta t^4 + \Delta t^{3/2})$  (Theorem 4.1 in [24]). The



$l^\infty(H^1)$ -estimate in [24] requires, for instance,  $\Delta t \leq h^{(3+2k)/8}$  for optimal convergence rate as  $h \rightarrow 0$ , but still predicts suboptimal convergence rate with respect to  $\Delta t \rightarrow 0$ .

The error estimates of Proposition 3.1, Theorem 3.5, 3.8, 3.10 give conditions in which  $p_h(\text{avg}) \in l^2(n_0, N; L^2)$ ,  $\mathbf{u}_h \in l^\infty(n_0, N; H^1)$ , and  $\partial_{\Delta t} \mathbf{u}_h \in l^2(L^2)$ . In particular, as a direct consequence of (36) and the conditions of Theorem 3.8, we have

$$\|\partial_{\Delta t} \mathbf{u}_h\|_{l^2(n_0, N; L^2)} + \nu \|\nabla \mathbf{u}_h\|_{l^\infty(n_0, N; L^2)} \leq C < \infty.$$

Estimates for pressure follow as well and are summarized in the next Corollary.

**Corollary 3.13.** *Under the conditions and conclusions of Theorem 3.8,*

$$\Delta t \sum_{n=n_0}^{N-1} \|p_h^{n+1/2}\| \leq C < \infty, \quad \text{as } h, \Delta t \rightarrow 0. \quad (39)$$

*Under the conditions and conclusions of Theorem 3.10, for  $s = k - 1$ ,*

$$\Delta t \sum_{n=n_0}^{N-1} \|p^{n+1/2} - p_h^{n+1/2}\| \leq C(h^k + \Delta t^2). \quad (40)$$

*Proof.* See Section 4.3. □

**3.1. Constant factors in convergence estimates.** For Proposition 3.1, define the spatial-modeling error in  $X_h \times Q_h$  by  $K_1$ , time modeling-error by  $K_2$ , and initial condition modeling error by  $K_0$  so that

$$K_0 := C + C\nu^{-1/2} \begin{cases} \|\mathbf{u}\|_{l^2(0, 2n_0; H^2)} & \text{if } n_0 \geq 1, \\ 0 & \text{otherwise} \end{cases} \quad (41)$$

$$K_1 := C\nu^{-1/2} \|\nabla \mathbf{u}\|_{l^\infty(L^2)} \|\nabla \mathbf{u}\|_{l^2(L^2)} + K_0 \|\mathbf{u}\|_{l^\infty(0, n_0; L^2)} + \dots + C\nu^{-1/2} \|p\|_{l^2(n_0, N; L^2)} + C\nu^{-1/2} h^2 \|\partial_t \mathbf{u}\|_{L^2(t^{n_0}, T; H^1)} \quad (42)$$

$$K_2 := C\nu^{-1/2} (\|\nabla \mathbf{u}\|_{l^\infty(L^2)} \|\nabla \mathbf{u}\|_{l^2(n_0, N; L^2)} + \dots + \|\partial_t \mathbf{u}\|_{L^2(t^{n_0}, T; W^{-1,2})}^2 + \|\partial_t \mathbf{u}\|_{l^2(n_0, N; W^{-1,2})}^2). \quad (43)$$

For Theorem 3.5 fix  $k > 0$ ,  $s > -1$ ,  $k^* = k$  (with  $k^* = 2$  if  $k = 1$ ) and define the weight of spatial modeling error in  $X_h$  by  $K_u$ , spatial modeling error in  $Q_h$  by  $K_p$ , and time modeling error by  $K_t$  so that

$$K_u := C(\nu^{-1/2} \|\nabla \mathbf{u}\|_{l^\infty(L^2)} \|\mathbf{u}\|_{l^2(H^{k+1})} + \nu^{-1/2} \|\partial_t \mathbf{u}\|_{L^2(t^{n_0}, T; H^{k^*-1})} + \dots + K_0 \|\mathbf{u}\|_{l^\infty(0, n_0; H^k)}) \quad (44)$$

$$K_p := C\nu^{-1/2} \|p\|_{l^2(n_0, N; H^{s+1})} \quad (45)$$

$$K_t := C\nu^{-1/2} (\|\partial_t^{(3)} \mathbf{u}\|_{L^2(t^{n_0}, T; W^{-1,2})} + \|\mathbf{u}\|_{l^\infty(H^2)} \|\partial_t^{(2)} \mathbf{u}\|_{L^2(L^2)} + \dots + \|\partial_t \mathbf{u}\|_{l^\infty(n_0, N; L^2)} \|\partial_t \mathbf{u}\|_{L^2(t^{n_0}, T; H^2)}) \quad (46)$$

$$\begin{aligned}
\overline{K}_u &:= C\Delta t^{1/2}\sigma(t^1)^{(k-1)/2}\nu^{-1/2}\|\nabla\mathbf{u}\|_{l^\infty(L^2)}\|\mathbf{u}^0\|_{k+1} + K_0\sigma(t^1)^{(k-1)/2}\|\mathbf{u}^0\|_k \\
&\quad + C\nu^{-1/2}\sigma(t^1)^{1/2}\|\nabla\mathbf{u}\|_{l^\infty(L^2)}(\Delta t\sum_{n=1}^N\sigma(t^n)^{k-2}\|\mathbf{u}^n\|_{k+1}^2)^{1/2} \\
&\quad + C\nu^{-1/2}(\int_{t^{n_0}}^T\sigma(t)^{k-1}\|\partial_t\mathbf{u}(\cdot,t)\|_{k-1}^2dt)^{1/2} \\
&\quad + C\sigma(t^1)^{1/2}(K_0\max_{1\leq n\leq n_0}\sigma(t^n)^{(k-2)/2}\|\mathbf{u}^n\|_k)
\end{aligned} \tag{47}$$

$$\overline{K}_p := C\nu^{-1/2}(\Delta t\sum_{n=n_0}^N\sigma(t^n)^{k-2}\|p^n\|_k^2)^{1/2} \tag{48}$$

$$\begin{aligned}
\overline{K}_t &:= C\nu^{-1/2}(\int_{t^{n_0}}^T\sigma(t)^2\|\partial_t^{(3)}\mathbf{u}(\cdot,t)\|_{-1}^2dt)^{1/2} \\
&\quad + C\nu^{-1/2}\sigma(t^{n_0})^{1/2}\|\mathbf{u}\|_{l^\infty(H^2)}(\int_{t^{n_0}}^T\sigma(t)\|\partial_t^{(2)}\mathbf{u}(\cdot,t)\|_2^2dt)^{1/2} \\
&\quad + C\nu^{-1/2}\sigma(t^{n_0})^{1/2}\Delta t\|\partial_t\mathbf{u}\|_{l^\infty(n_0,N;L^2)}(\int_{t^{n_0}}^T\sigma(t)\|\partial_t\mathbf{u}(\cdot,t)\|_2^2dt)^{1/2}.
\end{aligned} \tag{49}$$

Analogous to  $K_1$ ,  $K_2$ ,  $K_0$ , for Theorem 3.8 define the spatial-modeling error in  $X_h \times Q_h$  by  $F_1$ , time modeling-error by  $F_2$ , and initial condition modeling error by  $F_0$  so that

$$F_0 := C\nu^{1/2} + C \begin{cases} (\Delta t \sum_{n=0}^{2n_0-1} (\|\mathbf{u}^{n+1/2}\|_2^2 + h^{-1}|\mathbf{e}^{n+1/2}|_1^2))^{1/2} & \text{if } n_0 \geq 1, \\ 0 & \text{otherwise} \end{cases} \tag{50}$$

$$\begin{aligned}
F_1 &:= C\nu^{-1}\|\mathbf{u}\|_{l^\infty(n_0,N;H^2)}\|p\|_{l^2(L^2)} + \nu^{-1}F_0\|p\|_{l^\infty(0,n_0;L^2)} \\
&\quad + C\|\mathbf{u}\|_{l^\infty(n_0,N;H^2)}\|\nabla\mathbf{u}\|_{l^2(L^2)} + F_0\|\nabla\mathbf{u}\|_{l^\infty(0,n_0;L^2)} \\
&\quad + C\nu^{-1}h\|\partial_t p\|_{L^2(t^{n_0},T;L^2)} + Ch\|\partial_t\mathbf{u}\|_{L^2(t^{n_0},T;H^1)} \\
&\quad + C(\nu^{-1}h^{1/2}\|p\|_{l^\infty(H^1)} + \|\mathbf{u}\|_{l^\infty(H^2)})(\Delta t \sum_{n=n_0}^{N-1} |\mathbf{e}^{n+1/2}|_1^2)^{1/2}
\end{aligned} \tag{51}$$

$$\begin{aligned}
F_2 &:= C(\|\mathbf{u}\|_{l^\infty(H^2)}\|\partial_t\mathbf{u}\|_{L^2(t^{n_0},T;H^1)} + \|\partial_t^{(2)}\mathbf{u}\|_{L^2(t^{n_0},T;L^2)} + \dots \\
&\quad \dots + \nu\|\partial_t\mathbf{u}\|_{L^2(t^{n_0},T;H^2)} + \|\partial_t p\|_{L^2(t^{n_0},T;H^1)} + \|\partial_t \mathbf{f}\|_{L^2(t^{n_0},T;L^2)}).
\end{aligned} \tag{52}$$

For Theorem 3.10, fix  $k > 0$ ,  $s > -1$  and define the weight of spatial modeling error in  $X_h$  by  $F_u$ , spatial modeling error in  $Q_h$  by  $F_p$ , the bootstrapped modeling error of velocity in the energy norm by  $F_3$ , and time modeling error by  $F_t$  so that

$$\begin{aligned}
F_u &:= C(\|\mathbf{u}\|_{l^\infty(H^2)}\|\mathbf{u}\|_{l^2(H^{k+1})} + \|\partial_t\mathbf{u}\|_{L^2(t^{n_0},T;H^k)} + \dots \\
&\quad \dots + F_0\|\mathbf{u}\|_{l^\infty(0,n_0;H^{k+1})})
\end{aligned} \tag{53}$$

$$\begin{aligned}
F_p &:= C\nu^{-1}(\|\mathbf{u}\|_{l^\infty(H^2)}\|p\|_{l^2(H^{s+1})} + \|\partial_t p\|_{L^2(t^{n_0},T;H^s)} + \dots \\
&\quad \dots + F_0\|p\|_{l^\infty(0,n_0;H^{s+1})})
\end{aligned} \tag{54}$$

$$F_3 := C\|\mathbf{u}\|_{l^\infty(H^2)}(\|\mathbf{u}\|_{l^\infty(H^2)} + \nu^{-1}h^{1/2}\|p\|_{l^\infty(H^1)}) \tag{55}$$

$$\begin{aligned}
F_t &:= C(\|\partial_t^{(3)}\mathbf{u}\|_{L^2(t^{n_0},T;L^2)} + \|\mathbf{u}\|_{l^\infty(n_0,N;H^2)}\|\partial_t^{(2)}\mathbf{u}\|_{L^2(t^{n_0},T;H^1)} + \dots \\
&\quad \dots + \|\partial_t\mathbf{u}\|_{l^\infty(n_0,N;L^2)}\|\partial_t\mathbf{u}\|_{L^2(t^{n_0},T;H^3)})
\end{aligned} \tag{56}$$