Mathematik

Denny Otten

Spatial decay and spectral properties of rotating waves in parabolic systems

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Shaker Verlag
Aachen 2014

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at http://dnb.d-nb.de.

Zugl.: Bielefeld, Univ., Diss., 2014

Printed in Germany.

ISBN 978-3-8440-2663-4
ISSN 0945-0882

Shaker Verlag GmbH • P.O. BOX 101818 • D-52018 Aachen
Phone: 0049/2407/9596-0 • Telefax: 0049/2407/9596-9
Internet: www.shaker.de • e-mail: info@shaker.de

Spatial decay and spectral properties of rotating waves in parabolic systems

Der Fakultät für Mathematik der Universität Bielefeld eingereichte

Dissertation

zur Erlangung des akademischen Grades

Doktor-Mathematik (Dr.-Math.)

vorgelegt von

Dipl. Math. Denny Otten

Bielefeld, im Dezember 2013

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Dissertation eingereicht am: 17. Dezember 2013
Datum der mündlichen Prüfung: 07. Februar 2014

Im Zuge der Veröffentlichung wurde die vorliegende Dissertation redaktionell korrigiert.

Bielefeld, 20. Februar 2014

Contents

1 Introduction and main result

1.1 Introduction

The field of nonlinear waves has been extended over the last decades. Nonlinear waves are solutions of time dependent partial differential equations that are posed on an unbounded domain, [35, Chapter 18]. In many cases these equations possess symmetry properties which, depending on their type, allow traveling waves, rotating waves or phase-rotating waves. A common feature to all these solutions is that they are completely characterized by a time independent profile which travels, rotates or oscillates at constant velocity. Such solutions arise in different applications from physical, chemical and biological sciences. Equations that exhibit these types of solutions are for instance the complex Ginzburg-Landau equation (see: [64], [76]), the λ-ω system (see: [61], [80]), the Barkley model (see: [10], [11]), the Schrödinger equation (see: [33], [112]) and the Gross-Pitaevskii equation (see: [44]). One important focus of research is to study nonlinear stability of such solutions and relate it to spectral properties of the linearization at the nonlinear wave. For the numerical approximation it is crucial to study truncations to bounded domains. Proving exponential decay of waves is an important issue in this field, since it implies exponentially small truncation errors. This is one major step before investigating further errors caused by spatial and temporal discretizations.

In the present thesis we deal with systems of reaction-diffusion equations

$$(1.1)\qquad \begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0) &= u_0(x) \qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d, \end{aligned}$$

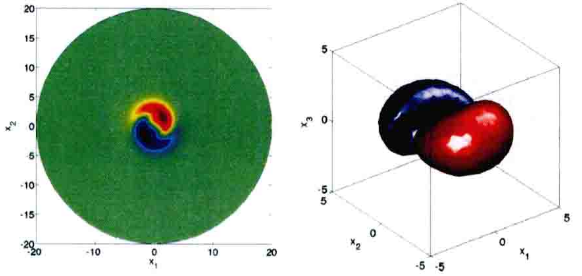
where $A\in\mathbb{R}^{N,N}$ is a diffusion matrix, $f:\mathbb{R}^N\to\mathbb{R}^N$ is a sufficiently smooth nonlinearity, $u_0:\mathbb{R}^d\to\mathbb{R}^N$ are the initial data and $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^N$ denotes a vector-valued solution which is sought for.

We are mainly interested in rotating wave solutions of (1.1) which are of the form

$$(1.2)\qquad u_\star(x,t) = v_\star(e^{-tS}x),\ t\geqslant 0,\ x\in\mathbb{R}^d,\ d\geqslant 2$$

with space-dependent profile $v_\star:\mathbb{R}^d\to\mathbb{R}^N$ and skew-symmetric matrix $S\in\mathbb{R}^{d,d}$.

As an example we discuss in this work the cubic-quintic complex Ginzburg-Landau equation (QCGL), cf. (2.1), where such solutions occur and are called spinning solitons. For more information on spinning solitons see [28]. Figure 1.1(a) shows the real part of a spinning soliton $v_\star$ in two space dimensions. The range of colorbar reaches from -1.6 (blue) to 1.6 (red). Figure 1.1(b) shows the isosurfaces of the real part of a spinning soliton in three space dimensions. The isosurfaces have the values -0.5 (blue) and 0.5 (red).



(a) (b)

Figure 1.1: Spinning soliton of QCGL (2.1) for $d = 2$ (a) and $d = 3$ (b)

Rotating waves from (1.2) are completely characterized by their time invariant profile $v_\star$ and a skew-symmetric matrix $S \in \mathbb{R}^{d,d}$. The skew-symmetry of S implies that e^{-tS} is a rotation matrix. Therefore, such a solution $u_\star$ rotates at constant velocity while it maintains its shape. Note that rotating waves always come in families: If $u_\star$ from (1.2) solves (1.1), then so does the function $v_\star(e^{-tS}(R^{-1}(x - \tau)))$ for every $(R, \tau) \in \mathrm{SE}(d)$, where $\mathrm{SE}(d)$ denotes the special Euclidean group. Furthermore, the profile $v_\star$ is called localized, if it tends to some constant vector $v_\infty \in \mathbb{R}^N$ as $|x| \to \infty$, and nonlocalized otherwise.

Transforming (1.1) via $u(x,t) = v(e^{-tS}x, t)$ into a co-rotating frame we obtain the evolution equation

$$
\begin{aligned}
v_t(x,t) &= A\triangle v(x,t) + \langle Sx, \nabla v(x,t)\rangle + f(v(x,t)),\ t > 0,\ x \in \mathbb{R}^d,\ d \geqslant 2, \\
v(x,0) &= u_0(x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad ,\ t = 0,\ x \in \mathbb{R}^d,
\end{aligned} \tag{1.3}
$$

with drift term

$$
\langle Sx, \nabla v(x)\rangle := \sum_{i=1}^{d} (Sx)_i D_i v(x). \tag{1.4}
$$

Now, the pattern $v_\star$ itself is a stationary solution of (1.3), meaning that $v_\star$ solves the steady state problem

$$
A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2, \tag{1.5}
$$

that involves the Ornstein-Uhlenbeck operator

$$
[\mathcal{L}_0 v](x) := A\triangle v(x) + \langle Sx, \nabla v(x)\rangle,\ x \in \mathbb{R}^d. \tag{1.6}
$$

An important issue is to investigate the *nonlinear stability* (also called *stability with asymptotic phase*) *of rotating waves*, i.e. to show that for any initial data u_0 sufficiently close to $v_\star$ there exists $(R_\infty, \tau_\infty), (R(t), \tau(t)) \in \mathrm{SE}(d)$ such that the solution $u(t)$, $t \geqslant 0$, of (1.1) satisfies $u(t) - v_\star(e^{-tS}(R(t)^{-1}(x - \tau(t)))) \to 0$ in a suitable topology and $(R(t), \tau(t)) \to (R_\infty, \tau_\infty)$ as $t \to \infty$. A well known task is to derive nonlinear stability from linear stability of the linearized operator

$$
[\mathcal{L}v](x) := [\mathcal{L}_0 v](x) + Df(v_\star(x))\, v(x),\ x \in \mathbb{R}^d. \tag{1.7}
$$

By *Linear stability* (this will be called *strong spectral stability* in Chapter 9) we mean that the essential spectrum and the isolated eigenvalues of $\mathcal{L}$ lie strictly to the left of the imaginary axis except those that are caused by the SE(d)-group action (see Chapter 9 for these eigenvalues). A nonlinear stability result for two dimensional localized rotating patterns was proved by Beyn and Lorenz in [15]. Their proof requires three essential assumptions: The profile $v_\star$ of the rotating wave and their partial derivatives up to order 2 are localized in the above sense. Furthermore, the matrix $Df(v_\infty)$ is stable, meaning that all its eigenvalues have a negative real part. And finally, strong spectral stability in the sense above is assumed. Their result shows that the decay of the rotating wave itself and the spectrum of the linearization are both crucial for investigating nonlinear stability of localized rotating waves. A corresponding result on nonlinear stability of nonlocalized rotating waves, such as spiral waves and scroll waves, is still an open problem. However, the spectrum of the linearization at a spiral wave is well-known and has been extensively studied by Sandstede, Scheel and Fiedler in [92], [38] and [93].

For numerical computations it is essential to truncate equation (1.1) and (1.3) to bounded domains, see Section 1.6. This is motivated by the fact that numerical approximations, e.g. with finite elements, require that the original equation is posed on a bounded domain. The truncation error, that arises by the truncation process, depends on the boundary conditions. Assuming that a rotating wave is (exponentially) localized, we can expect the truncation error to be (exponentially) small as well. For this reason, the *exponential decay of rotating waves* plays a fundamental role in the field of *truncations* and *approximations of rotating waves on bounded domains*.

The basic step before investigating truncations is to study the rotating waves of (1.1) on the whole $\mathbb{R}^d$. This is the topic of the present thesis. For the behavior on bounded domains there are a lot of numerical simulations but the analysis of the limit as $R \to \infty$ is an open problem, see Section 1.6.

The main theme of this work is to derive suitable conditions guaranteeing that every localized rotating wave of (1.1) is already exponentially localized. To be more precise, the main theorem states that every rotating wave that falls below a certain threshold at infinity and that satisfies in addition $v_\star \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ for some $1 < p < \infty$, decays exponentially in space, in the sense that $v_\star$ belongs to some exponentially weighted Sobolev space $W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$. Afterward, we extend this result to complex-valued systems. This is motivated by the exponentially localized spinning solitons arising in the complex Ginzburg-Landau equation, see Figure 1.1.

We follow Mielke and Zelik, [114], and define the exponentially weighted Sobolev spaces $W^{k,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$ for some weight function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ of exponential growth rate. The main suggestion for our result comes from [15]. In [15, Remark 5], the authors conjecture that the stability of the matrix $Df(v_\infty)$, i.e. $\operatorname{Re}\sigma\left(Df(v_\infty)\right) < 0$, implies the exponential decay of the rotating wave as $|x| \to \infty$. Assuming in addition that $v_\star$ is localized, they believe that one can also deduce that its partial derivatives up to order 2 are localized. For traveling waves in dimension $d = 1$ such results are well known. There one usually considers x as the time variable, transforms the steady state problem to a first order ODE and applies the theory of exponential dichotomies. But the procedure does not carry over directly to $d \geqslant 2$.

Therefore, we develop in this thesis a new approach that allows to prove exponential decay in higher space dimensions.

Our approach works as follows: In the first step we compute a (complex-valued) heat kernel H_0 for the differential operator $\mathcal{L}_0$. Using this kernel, we introduce the associated semigroup $(T_0(t))_{t\geqslant 0}$ on an appropriate state space X, e.g. $X = L^p(\mathbb{R}^d, \mathbb{C}^N)$, $X = C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{C}^N)$ or $X = C^\alpha(\mathbb{R}^d, \mathbb{C}^N)$. We verify that the semigroup is strongly continuous on X (or possibly on a certain subspace of X), which justifies to introduce the infinitesimal generator and to apply semigroup theory, see e.g. Engel and Nagel, [34]. The generator itself can be considered as the abstract version of the formal differential operator $\mathcal{L}_0$. To investigate their relation we must solve the identification problem, which on the one hand yields an explicit representation for the maximal domain and on the other hand shows that the abstract and the formal differential operator coincide on this domain. The identification problem was solved for the scalar real-valued case by Metafune, Pallara and Vespri in [73] for $X = L^p(\mathbb{R}^d, \mathbb{R})$ and by Da Prato and Lunardi in [29] for $X = C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R})$ and $X = C^\alpha(\mathbb{R}^d, \mathbb{R})$.

For investigating the asymptotic behavior of solutions of (1.5) we decompose $Df(v_\star(x))$ as follows

$$Df(v_\star(x)) = Df(v_\infty) + Q_\varepsilon(x) + Q_c(x),\ x \in \mathbb{R}^d, \tag{1.8}$$

for some small perturbation $Q_\varepsilon \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$ and for some perturbation $Q_c \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$ with compact support. Then we show that it suffices to analyze solutions of the linear operator

$$[\mathcal{L}_0 v](x) + (Df(v_\infty) + Q_\varepsilon(x) + Q_c(x))\, v(x) = 0,\ x \in \mathbb{R}^d.$$

For this reason, we apply semigroup theory to study constant coefficient perturbations as well as small and compactly supported variable coefficient perturbations of $\mathcal{L}_0$.

We are faced with different problems in this work: The main problem is that the rotational term $\langle Sx, \nabla v_\star(x)\rangle$ has unbounded coefficients. Therefore, this term cannot be treated as a lower order term on unbounded domains. Moreover, since we consider complex-valued systems, we have to transfer many results, that are only known for the scalar real-valued case, to complex systems. Furthermore, due to the unbounded coefficients of $\langle Sx, \nabla v_\star(x)\rangle$ it turns out to be hard to solve the identification problem for $\mathcal{L}_0$. And finally, there is the question about a suitable state space X.

Furthermore, we investigate the eigenvalue problem for the linearization (1.7) at a localized rotating wave $v_\star$. We determine the eigenvalues located on the imaginary axis and caused by the SE(d)-group action as follows

$$\sigma(S) \cup \{\lambda_1 + \lambda_2 \mid \lambda_1, \lambda_2 \in \sigma(S),\ \lambda_1 \neq \lambda_2\} \subseteq \sigma_{\mathrm{point}}(\mathcal{L}). \tag{1.9}$$

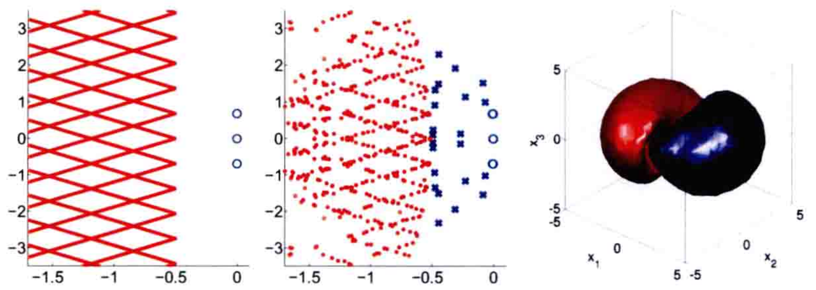
And we derive the shape of the corresponding eigenfunctions $v : \mathbb{R}^d \to \mathbb{C}^N$, which are of the form

$$v(x) = \left\langle C^{\mathrm{rot}} x + C^{\mathrm{tra}}, \nabla v_\star(x)\right\rangle,\ x \in \mathbb{R}^d,$$

for some explicitly given skew-symmetric $C^{\text{rot}} \in \mathbb{C}^{d,d}$ and $C^{\text{tra}} \in \mathbb{C}^{d}$, where $\langle \cdot, \cdot \rangle$ is defined as in (1.4). In particular, the result shows that for every space dimension $d \geqslant 2$ the eigenvalue $\lambda = 0$ belongs to $\sigma_{\text{point}}(\mathcal{L})$ with associated eigenfunction $v(x) = \langle Sx, \nabla v_\star(x) \rangle$. Another application of our main theorem shows that eigenfunctions of the linearized operator decay exponentially in space, provided the corresponding eigenvalues are sufficiently close to the imaginary axis. In addition to eigenvalues, we identify a certain part of the essential spectrum,

(1.10)
$$\left\{ -\lambda(\omega) - i \sum_{l=1}^{k} n_l \sigma_l \mid n_l \in \mathbb{Z},\ \lambda(\omega) \in \sigma(\omega^2 A - Df(v_\infty)),\ \omega \in \mathbb{R} \right\} \subseteq \sigma_{\text{ess}}(\mathcal{L}),$$

where $\pm i\sigma_1, \ldots, \pm i\sigma_k$ denote the nonzero eigenvalues of S. For this purpose we derive a dispersion relation for localized rotating patterns. All these studies are motivated by [15] and [71] and are necessary to investigate nonlinear stability of rotating waves in higher space dimensions.

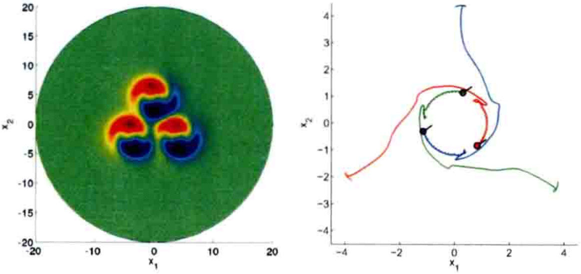


(a) (b) (c)

Figure 1.2: Essential and point spectrum (a), numerical spectrum (b) and two isofurfaces of eigenfunction corresponding to the eigenvalue 0 (c) for the three-dimensional spinning soliton of QCGL (2.1) from Figure 1.1(b)

Figure 1.2(a)–1.2(b) illustrates the spectral behavior of the QCGL when linearized at the spinning soliton from Figure 1.1(b). The red lines in Figure 1.2(a) correspond to the part of the essential spectrum from (1.10). They form a zig-zag structure that is parallel to the imaginary axis. The distance of two neighboring tips of the cones equals the rotational velocity $\sigma_1 = 0.68576$. The blue circles correspond to the part of the point spectrum from (1.9), that is caused by the SE(3)-group action. Each of these isolated eigenvalues has multiplicity 2. Figure 1.2(b) shows a numerical approximation of the full spectrum. Red dots approximate the essential spectrum, blue circles the known eigenvalues on the imaginary axis, and blue crosses the remaining point spectrum. Figure 1.2(c) illustrates an approximation of the eigenfunction $\langle Sx, \nabla v_\star(x) \rangle$ that corresponds to the zero eigenvalue. The isosurfaces have values -1.5 (blue) and 1.5 (red). Note that the eigenfunction coincides with our drift term and decays exponentially in space.

Finally, we numerically investigate the interaction of several spinning solitons in the cubic-quintic complex Ginzburg-Landau equation in two space dimensions. We are mainly interested in the fate of the single shapes and velocities when solitons collide or repel each other. In order to analyze the interaction of multi-solitons we extend the decompose and freeze method from Beyn, Thümmler and Selle, [17], to higher space dimensions. It writes the solution of (1.1) as a superposition of finite number of solutions (given by the number of patterns) which solve a system of coupled nonlinear partial differential algebraic equations.



(a) (b)

Figure 1.3: Interaction of three spinning solitons in the QCGL (2.1) with $d = 2$ and their positions of centers

Figure 1.3(a) shows the real part of the sum of three spinning solitons of the QCGL for $d = 2$, cf. Figure 1.1(a). Each of these solitons is located on a different vertex of an equilateral triangle and rotates at constant velocity. After some time they collide into a single spinning soliton that rotates at their common velocity. Figure 1.3(b) shows the time evolution for the positions of the 3 spinning solitons, that are obtained from the decompose and freeze method for multi-solitons. Each of the colors represent the motion of a single soliton with a pointer at the end which indicates the current phase position. For a detailed description we refer to Section 10.6.

1.2 Assumptions and main result

Below we give a more technical outline of the basic assumptions and the main result of this thesis:

Consider the steady state problem of the form

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2, \tag{1.11}$$

with **diffusion matrix** $A \in \mathbb{K}^{N,N}$ and a function $f : \mathbb{K}^N \to \mathbb{K}^N$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The **drift term** is defined by a matrix $0 \neq S \in \mathbb{R}^{d,d}$ as

$$\langle Sx, \nabla v(x)\rangle := \sum_{i=1}^{d}\sum_{j=1}^{d} S_{ij}x_j D_i v(x). \tag{1.12}$$

where $D_i = \frac{\partial}{\partial x_i}$. The operator $A\triangle v(x) + \langle Sx, \nabla v(x)\rangle$ is usually called the **complex Ornstein-Uhlenbeck operator**, [107].

Our interest is in skew-symmetric matrices $S = -S^T$, in which case (1.12) is a **rotational term** containing angular derivatives

$$\langle Sx, \nabla v(x)\rangle = \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} S_{ij}\left(x_j D_i - x_i D_j\right) v(x). \tag{1.13}$$

We look for different types of solutions, which satisfy at least $v \in L^p(\mathbb{R}^d, \mathbb{K}^N)$ for some $1 \leqslant p \leqslant \infty$ and $N \in \mathbb{N}$.

Under appropriate conditions our main result states that a solution $v_\star$ of (1.11) and its first order derivatives decay exponentially in space as the radius $|x|$ goes to infinity.

Investigating steady state problems of this type is motivated by the stability theory of rotating patterns in several spatial dimensions, [15]. There one considers reaction diffusion equations

$$\begin{aligned} u_t(x,t) =& A\triangle u(x,t) + f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0) =& u_0(x) \qquad\qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d, \end{aligned} \tag{1.14}$$

where $A \in \mathbb{K}^{N,N}$ is a diffusion matrix, $f : \mathbb{K}^N \to \mathbb{K}^N$ a nonlinearity and u a solution that maps $\mathbb{R}^d \times [0, \infty[$ into $\mathbb{K}^N$.

We define a rotating wave solution $u_\star$ of (1.14) in the following sense:

Definition 1.1. A function $u_\star : \mathbb{R}^d \times [0,\infty[\to \mathbb{K}^N$ is called a **rotating wave** (or **rotating pattern**) if it has the form

$$u_\star(x,t) = v_\star(e^{-tS}(x - x_\star)),\ x\in\mathbb{R}^d,\ t\in[0,\infty[, \tag{1.15}$$

with **profile** (or **pattern**) $v_\star : \mathbb{R}^d \to \mathbb{K}^N$, a skew-symmetric matrix $0 \neq S \in \mathbb{R}^{d,d}$ and $x_\star \in \mathbb{R}^d$. A rotating wave $u_\star$ satisfying

$$\lim_{|x|\to\infty} e^{\eta|x|}\left|v_\star(x) - v_\infty\right| = 0 \text{ for some } v_\infty \in \mathbb{K}^N \text{ and } \eta \geqslant 0$$

for $\eta = 0$ is called **localized** and **nonlocalized**, otherwise. Moreover, a localized rotating wave $u_\star$ is called **exponentially localized (with decay rate η)** if $\eta > 0$.

The vector $x_\star \in \mathbb{R}^d$ can be considered as the center of rotation for $d = 2$ and as the support vector of the axis of rotation for $d = 3$. In case $d \in \{2,3\}$, S can be considered as the angular velocity tensor associated to the angular velocity vector $\omega \in \mathbb{R}^{\frac{d(d-1)}{2}}$ containing S_{ij}, $i = 1,\ldots,d-1$, $j = i+1,\ldots,d$.

A transformation into a **co-rotating frame** shows that if $u(x,t)$ solves (1.14) then $v(x,t) = u(e^{tS}x + x_\star, t)$ solves

$$\begin{aligned} v_t(x,t) =& A\triangle v(x,t) + \langle Sx, \nabla v(x,t)\rangle + f(v(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ v(x,0) =& u_0(x) \qquad\qquad\qquad\qquad\qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d, \end{aligned} \tag{1.16}$$

where the drift term is given by (1.13). Conversely, if $v(x,t)$ solves (1.16) then $u(x,t) = v(e^{-tS}(x - x_\star), t)$ solves (1.14).

Note that $v_\star$ is a stationary solution of (1.16), meaning that $v_\star$ solves the nonlinear problem (1.11). In Section 2.1 we illustrate such rotating patterns by a series of examples.

In order to investigate exponential decay of the profile $v_\star$, we list a series of assumptions that will be important in the sequel. Throughout, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$:

Assumption 1.2. *For $A \in \mathbb{K}^{N,N}$ consider the following conditions:*

(A1) *A is diagonalizable (over $\mathbb{C}$), (system condition)*

(A2) $\operatorname{Re}\sigma(A) > 0$ *(ellipticity condition)*
where $\sigma(A)$ denotes the spectrum of A,

(A3) $\exists\, \beta_A > 0 : \operatorname{Re}\langle w, Aw\rangle \geqslant \beta_A \;\forall\, w \in \mathbb{K}^N,\ |w| = 1,$ *(accretivity condition)*
where $\langle u, v\rangle := \overline{u}^T v$ denotes the standard inner product on $\mathbb{K}^N$,

(A4) *case $(N = 1,\ \mathbb{K} = \mathbb{R})$: $A = a > 0$,*
cases $(N \geqslant 2,\ \mathbb{K} = \mathbb{R})$ and $(N \geqslant 1,\ \mathbb{K} = \mathbb{C})$:
$\mu_1(A) > \dfrac{|p-2|}{p}$ *for some fixed $1 < p < \infty$ (L^p-antieigenvalue condition)*
where $\mu_1(A)$ is the first antieigenvalue of A.

The assumptions (A1)–(A4) satisfy the obvious relations:

$$(A4) \Rightarrow (A3) \Rightarrow (A2).$$

Condition (A1) ensures that all results for scalar equations can be extended to system cases. It is completely independent of (A2)–(A4). Assumption (A2) guarantees that the diffusion part $A\triangle$ is an elliptic operator and requires that all eigenvalues λ of A are contained in the open right half-plane $\mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$, where $\sigma(A)$ denotes the spectrum of A. A matrix $C \in \mathbb{K}^{N,N}$ that satisfies $\operatorname{Re}\sigma(C) < 0$ is called a **stable matrix**. Thus, (A2) states that the matrix $-A$ is stable. In particular, (A2) implies that the matrix A is invertible. Condition (A3), states that A is an **strongly accretive matrix**, which is more restrictive than (A2). Assumption (A4) postulates that the **first antieigenvalue of** A, defined by, [48],

$$\mu_1(A) := \inf_{\substack{w \in \mathbb{K}^N \\ w \neq 0 \\ Aw \neq 0}} \frac{\operatorname{Re}\langle w, Aw\rangle}{|w||Aw|} = \inf_{\substack{w \in \mathbb{K}^N \\ |w| = 1 \\ Aw \neq 0}} \frac{\operatorname{Re}\langle w, Aw\rangle}{|Aw|},$$

is bounded from below by a non-negative p-dependent constant. This is equivalent to the following p-dependent upper bound for the (**real**) **angle of** A, [47],

$$\Phi_{\mathbb{R}}(A) := \cos^{-1}(\mu_1(A)) < \cos^{-1}\left(\frac{|p-2|}{p}\right) \in]0, \frac{\pi}{2}], \quad 1 < p < \infty.$$

Condition (A4) imposes additional requirements on the spectrum of A and is more restrictive than (A3). For some special cases, the constant $\mu_1(A)$ can be given explicitly in terms of the eigenvalues of A. In the scalar complex case $A = \alpha \in \mathbb{C}$, assumption (A4) leads to a cone condition which requires α to lie in a p-dependent sector in the right half-plane. In the scalar case condition (A4) coincides with the L^p-dissipativity condition from [26].

Assumption 1.3. *The matrix* $S \in \mathbb{R}^{d,d}$ *satisfies*

(A5) *S is skew-symmetric, i.e.* $S = -S^T$, $S \in \mathfrak{so}(d, \mathbb{R})$ *(rotational condition).*

Assumption (A5) guarantees that the drift term (1.12) contains only angular derivatives, see (1.13). Our main result will be formulated for the real-valued case.

Assumption 1.4. *The function* $f : \mathbb{R}^N \to \mathbb{R}^N$ *satisfies*

(A6) $f \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ *(smoothness condition).*

Later on we apply our results also to complex-valued nonlinearities of the form

$$f : \mathbb{C}^N \to \mathbb{C}^N, \quad f(u) = g\left(|u|^2\right) u,$$

where $g : \mathbb{R} \to \mathbb{C}^{N,N}$ is a sufficiently smooth function. Such nonlinearities arise for example in Ginzburg-Landau equations, Schrödinger equations, $\lambda - \omega$ systems and many other equations from physical sciences, see Section 2.1. Note, that in this case, the function f is not holomorphic in $\mathbb{C}$, but its real-valued version in $\mathbb{R}^2$ satisfies (A6) if g is in C^2. For differentiable functions $f : \mathbb{R}^N \to \mathbb{R}^N$, Df denotes the Jacobian matrix in the real sense, see the following conditions (A8) and (A9).

Assumption 1.5. *For* $v_\infty \in \mathbb{R}^N$ *consider the following conditions:*

(A7) $f(v_\infty) = 0$ *(constant asymptotic state),*

(A8) $A, Df(v_\infty) \in \mathbb{R}^{N,N}$ *are simultaneously diagonalizable (over* $\mathbb{C}$*) (system condition),*

(A9) $\sigma\left(Df(v_\infty)\right) \subset \mathbb{C}_- := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\}$ *(spectral condition).*

Condition (A7) states that v_∞ is a zero of the nonlinearity f. Note, that by (A8) assumption (A1) is automatically satisfied. Condition (A9) states that the matrix $Df(v_\infty)$ is stable.

Definition 1.6. A function $v_\star : \mathbb{R}^d \to \mathbb{K}^N$ is called a **classical solution of** (1.11) if

(1.17) $$v_\star \in C^2(\mathbb{R}^d, \mathbb{K}^N) \cap C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{K}^N)$$

and $v_\star$ solves (1.11) pointwise.

Equation (1.17) requires $v_\star$ to be C^2-smooth and bounded, see Section 3.2 for general function spaces. For a matrix $C \in \mathbb{K}^{N,N}$ we denote by $\sigma(C)$ the **spectrum of** C, by $\rho(C) := \max_{\lambda\in\sigma(C)} |\lambda|$ the **spectral radius of** C and by $s(C) := \max_{\lambda\in\sigma(C)} \operatorname{Re}\lambda$ the **spectral abscissa** (or **spectral bound**) **of** C. Using this notation, we define the constants

(1.18) $$a_{\min} := \left(\rho\left(A^{-1}\right)\right)^{-1}, \quad a_0 := -s(-A),$$
$$a_{\max} := \rho(A), \qquad b_0 := -s(Df(v_\infty)).$$

Our main tool for investigating exponential decay in space are exponentially weighted function spaces, which we introduce in Section 3 in detail. An essential ingredient for these function spaces is the choice of the weight function, which follows [114, Def. 3.1]:

Definition 1.7. (**1**) A function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ is called a **weight function of exponential growth rate** $\eta \geqslant 0$ provided that

$$\text{(W1)} \qquad \theta(x) > 0 \,\forall\, x \in \mathbb{R}^d,$$

$$\text{(W2)} \qquad \exists\, C_\theta > 0 :\; \theta(x+y) \leqslant C_\theta \theta(x) e^{\eta|y|} \,\forall\, x, y \in \mathbb{R}^d.$$

(**2**) A weight function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ of exponential growth rate $\eta \geqslant 0$ is called **radial** provided that

$$\text{(W3)} \qquad \exists\, \phi : [0, \infty[\to \mathbb{R} :\; \theta(x) = \phi(|x|) \,\forall\, x \in \mathbb{R}^d.$$

(**3**) A radial weight function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ of exponential growth rate $\eta \geqslant 0$ is called **non-decreasing** (or **monotonically increasing**) provided that

$$\text{(W4)} \qquad \theta(x) \leqslant \theta(y) \,\forall\, x, y \in \mathbb{R}^d \text{ with } |x| \leqslant |y|.$$

Note, that radial weight functions satisfy $\theta(x) = \theta(y)$ for every $x, y \in \mathbb{R}^d$ with $|x| = |y|$. Standard examples are

$$\theta_1(x) = \exp(-\mu|x|) \quad \text{and} \quad \theta_2(x) = \cosh(\mu|x|),$$

as well as their smooth analogs

$$\theta_3(x) = \exp\left(-\mu\sqrt{|x|^2+1}\right) \quad \text{and} \quad \theta_4(x) = \cosh\left(\mu\sqrt{|x|^2+1}\right),$$

for $\mu \in \mathbb{R}$ and $x \in \mathbb{R}^d$. Obviously, all these functions are radial weight functions of exponential growth rate $\eta = |\mu|$ with $C_\theta = 1$. Moreover, θ_1, θ_3 are non-decreasing and θ_2, θ_4 are non-decreasing if $\mu \leqslant 0$. Note, that for $\mu = 0$ the examples include the weight function $\theta(x) = 1$. Furthermore, Definition 1.7 includes (radial) tableau functions, e.g.

$$\theta_5(x) = \begin{cases} 1 & , |x| \leqslant R, \\ \exp(-\mu(|x|-R)) & , |x| \geqslant R, \end{cases}$$

for some $R > 0$, where the constant C_θ depends on the size of the support, but not on the growth rate η.

Associated with weight functions of exponential growth rate are **exponentially weighted Lebesgue** and **Sobolev spaces**

$$\begin{aligned} L^p_\theta(\mathbb{R}^d, \mathbb{K}^N) :=& \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{K}^N) \mid \|\theta u\|_{L^p} < \infty\}, \\ W^{k,p}_\theta(\mathbb{R}^d, \mathbb{K}^N) :=& \{u \in L^p_\theta(\mathbb{R}^d, \mathbb{K}^N) \mid D^\beta u \in L^p_\theta(\mathbb{R}^d, \mathbb{K}^N) \,\forall\, |\beta| \leqslant k\}, \end{aligned}$$

for every $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{N}_0$. Our main result is the following:

Theorem 1.8 (Exponential decay of $v_\star$). *Let the assumptions* (A4)–(A9) *be satisfied for some* $1 < p < \infty$ *and* $\mathbb{K} = \mathbb{R}$. *Then for every* $0 < \vartheta < 1$ *and for every radially nondecreasing weight function* $\theta \in C(\mathbb{R}^d, \mathbb{R})$ *of exponential growth rate* $\eta \geqslant 0$ *with*

$$0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3} \frac{a_0 b_0}{a_{\max}^2 p^2}$$