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Wolfgang Lenski (Ed.)

Logic versus Approximation

Essays Dedicated to Michael M. Richter on the Occasion of his 65th Birthday

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Preface

There are prominent moments in time when ongoing scientific work is interrupted for a moment and a more general perspective is sought. The symposium on *Logic Versus Approximation* was one such moment, a chance to reflect on the influences of the scientific work of Michael M. Richter on the occasion of his 65th birthday. This collection is a selection of contributions to this symposium.

In focussing on today's knowledge-based systems we encounter two major paradigms of reasoning. There are on the one hand the logic-based approaches where 'logic' is to be understood in a rather broad sense. This approach is predominantly deployed in symbolic domains where numerical calculations are not the core challenge. Logic has without any doubt provided a powerful methodological tool. Progress in this area is mainly performed by refining the representation of structural aspects which results in a succession of models that should capture increasingly more aspects of the domain. This may be seen as an approximation process on the metalevel.

There is also some weakness in the logic-based approach, though, which is due to its very foundation. The semantic theory of truth by Tarski has explicitly eliminated personal influence on the validity of truth as well as the representation of dynamically changing variations of the ground terms inside the theory. It does not allow for adaptive individual behavior per se as is, for example, explicitly required in the field of e-commerce. From a methodological point of view *pragmatics* is required as opposed to *semantics.* These aspects make it worth considering rather philosophical and foundational reflections as well.

On the other hand we find approximation-oriented reasoning. Methods of this kind are mainly applied in numerical domains where approximation is part of the scientific methodology itself. Here we again distinguish two different basic types, discrete and continuous domains.

However, from a more abstract level all these approaches do focus on similar topics and arise on various levels such as problem modeling, inference and problem-solving mechanisms, algorithms and mathematical methods, and mathematical relations between discrete and continuous properties, and are integrated in tools and applications.

Research on both kinds of reasoning in these areas has mostly been conducted independently so far. Whereas approaches based on discrete or continuous domains influence each other in a sometimes surprising way, influences between these and the symbolic approach have been less intensively studied. Especially the potentialities of an integration are certainly not understood to a satisfactory degree although the primary focus from an abstract point of view is on a similar topic. It requires a unifying vision to which all parts have to contribute from their own perspectives.

Scientific work is necessarily always the construction of sense. Progress is by no means arbitrary, but always guided by a quest for a still better understanding of parts of the world. Such construction processes are essentially hermeneutical ones, and an emanating coherent understanding of isolated topics is only guaranteed through a unifying view of a personal vision. Such a vision is especially provided by the research interests of Michael M. Richter, which have influenced the overall perspective of the symposium. In this sense his scientific work was present all the time during the symposium.

Michael M. Richter has exerted a wide influence on logic and computer science. Although his productive work is widely spread there are some general interests behind them. A central interest is certainly in modelling structural aspects of reality for problem solving along with the search for adequate methodologies for this purpose. Michael M. Richter made significant contributions, however, to a wide variety of topics ranging from purely logical problems in model theory and non-standard analysis to representation techniques in computer science that have finally emancipated themselves from their logical origins with special emphasis given to problems in artificial intelligence and knowledge-based systems.

The symposium on *Logic Versus Approximation* brought insight into these different approaches and contributed to the emergence of a unifying perspective. At the same time it reflected the variety of Michael M. Richter's scientific interests. The contributions to this volume range from logical problems, philosophical considerations, applications of mathematics and computer science to real-world problems, and programming methodologies up to current challenges in expert systems.

The members of the organization and program committee especially wish to thank the authors for submitting their papers and responding to the feedback provided by the referees. We also wish to express our gratitude to the *FAW-Förderkreis e.V.* and the *empolis GmbH* for their valuable support. Finally, we are very grateful to the local organization team of the International Conference and Research Center for Computer Science at Schloß Dagstuhl for their professionalism in handling the local arrangements.

To honor Michael M. Richter, the President of the University of Kaiserslautern, Prof. Dr. Helmut J. Schmidt, in his diverting opening talk surveyed the creative powers of Michael M. Richter garnished with concise anecdotes on mutual personal experiences at Kaiserslautern University. To pay special tribute to the work of Michael M. Richter the scientific program of the symposium was complemented by lively and inspiring after-dinner speeches by Franz Josef Radermacher and Paul Stähly. The surroundings of the International Conference Center of Schloß Dagstuhl provided the appropriate atmosphere and greatly helped in making the symposium a scientifically intriguing, socially enjoyable, and altogether most memorable occasion. This collection is meant to capture the essence of its scientific aspects.

Kaiserslautern, December 2003 Wolfgang Lenski

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A True Unprovable Formula of Fuzzy Predicate Logic*

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Abstract. We construct a formula true in all models of the product fuzzy predicate logic over the standard product algebra on the unit real interval but unprovable in the product fuzzy logic (and hence having truth value less than 1 in some model over a non-standard linearly ordered product algebra). Gödel's construction of a true unprovable formula of arithmetic is heavily used.

1 Introduction

Is true the same as provable? For classical logic, Gödel's completeness theorem says that a formula φ is provable in a theory T (over classical logic) iff φ is true in all models of T. On the other hand, if "true" does not mean "true in all models of the theory" but "true in the intended (standard model", Gödel's first incompleteness theorem applies: if T is a recursively (computably) axiomatized arithmetic whose axioms are true in the structure $\mathbf{N}$ of natural numbers (the standard model of arithmetic) then there is a formula true in $\mathbf{N}$ but unprovable in T. Thus such arithmetic only *approximates* the truth.

Pure classical logic (or the empty theory T with no special axioms) over classical logic does not distinguish any standard non-standard models. It has just two truth values: true and false. For fuzzy logic the situation is different since one can distinguish standard and non-standard algebras of truth functions of connectives. Fuzzy logic is a many valued logic, the standard set of truth values being the unit real interval $[0,1]$. The truth values are ordered; one formula may be more true than another formula (comparative notion of truth). This formalizes truth of vague (imprecise) propositions (like "He is a tall man", "This is a very big number", "I shall come soon" etc.). Most often, fuzzy logic is truth-functional, i.e. works with truth functions of connectives. In particular, for basic fuzzy predicate logic $BL\forall$ (see [1]) each continuous t-norm together with its residuum determines a standard algebra of truth functions. A continuous t-norm is a continuous binary operation $*$ on the real unit square which is commutative, associative, non-decreasing in each argument, having 1 for its unit element

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($x * 1 = x$ for all x) and having 0 for its zero element ($x * 0 = 0$ for all x). Its residuum $\Rightarrow$ is defined as follows: $x \Rightarrow y = \max\{z | x * z \leq y\}$. Any continuous t-norm can serve as the truth function of conjunction; then its residuum is taken to be the truth function of implication. This is the standard semantics of the basic fuzzy logic.

General semantics is given by the class of BL-algebras. The definition is to be found e.g. in [1], see also below Sect.2 Roughly, a BL-algebra is a (lattice)-ordered structure with a least element and a greatest element and two operations $*, \Rightarrow$ that "behave like continuous t-norms". Each continuous t-norm with its residuum defines a standard BL-algebra. Given a predicate language $\mathcal{I}$ and BL-algebra $\mathbf{L}$, an $\mathbf{L}$-interpretation $\mathbf{M}$ of $\mathcal{I}$ consists of a crisp non-empty domain M and for each predicate P of an $\mathbf{L}$-fuzzy relation (of the respective arity) on $\mathbf{M}$.[1] One gives a natural definition of the truth value $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ (v being an evaluation of object variables by elements of $\mathbf{M}$) in Tarski's style by induction on the complexity of φ. For $\mathbf{L}$ being standard, $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ is always defined; for general $\mathbf{L}$ one has to work with so-called safe interpretations, for which $\|\varphi\|^{\mathbf{L}}_{\mathbf{M},v}$ is total (defined for all φ). (For standard $\mathbf{L}$ each interpretation is safe.) *Axioms* and *deduction rules* of $BL\forall$ can be found in [1]; this gives the notion of *provability* in $BL\forall$. The *general completeness theorem* says that a formula φ is provable in the basic fuzzy predicate logic $BL\forall$ iff it is true (= has value 1) for each safe interpretation $\mathbf{M}$ over each linearly ordered BL-algebra $\mathbf{L}$ and each evaluation v (φ is a *general BL-tautology*).

Call φ a *standard BL-tautology* if φ is true for each (safe) interpretation over each standard BL-algebra. And as Montagna showed [4], the set of all standard BL-tautologies is not arithmetical and hence is a proper subset of the set of all general tautologies. A simple example of a formula which is a standard BL-tautology but not a general BL-tautology is the formula

$$(\forall x)(\varphi \& \nu) \equiv [(\forall x)\varphi \& \nu] \qquad (B)$$

where x is not free in ν. (A counterexample was found by F. Bou.)

Now turn to the three well-known particular continuous t-norms, namely Gödel t-norm ($x * y = \min(x, y)$), Łukasiewicz t-norm ($x * y = \max(0, x + y - 1)$) and product t-norm ($x * y = x \cdot y$) and the corresponding fuzzy predicate logics $G\forall$, Ł$\forall$, $\Pi\forall$. See [1] for details; now we only mention that the unique standard algebra of truth functions of each of those logics is given by the corresponding t-norm whereas general algebras of truth functions are algebras from the variety generated by the standard algebra, so-called MV-algebras, Gödel algebras and product algebras respectively. Axioms are those of $BL\forall$ extended by the axiom of idempotence of conjunction $\varphi \to (\varphi \& \varphi)$ for Gödel logic, the axiom of double negation ($\neg\neg\varphi \to \varphi$ where $\neg\varphi$ is $\varphi \to \bar{0}$) for Łukasiewicz logic and by two axioms presented below for product logic. All three logics have general completeness theorem (for formulas true in all safe interpretations over all linearly ordered

[1] This means that if P is n-ary then its interpretation is a mapping assigning to each n-tuple of elements of $\mathbf{M}$ an element of $\mathbf{L}$ – the degree of membership of the tuple into the interpreting fuzzy relation.

respective algebras); for Gödel logic standard tautologies coincide with general tautologies, for Łukasiewicz logic the set of standard tautologies is Π_2-complete (Ragaz), for product logic the set of standard tautologies is not arithmetical (Montagna). All three logics prove the formula (B) above (for $G\forall$ and Ł$\forall$ see [1], for $\Pi\forall$ see below). The aim of the present paper is to exhibit a particular example of a formula being a standard but not general tautology of $\Pi\forall$ (thus an unprovable standard tautology of $\Pi\forall$). As mentioned above, our formula will be a variant of Gödel's famous self-referential formula stating its own unprovability. But keep in mind that we are interested in pure (product) logic, not in a theory over this logic. To find such an example for Łukasiewicz predicate logic remains an open problem.

2 The Product Predicate Logic

Recall that a BL-algebra is a residuated lattice $\mathbf{L} = (L, \wedge, \vee, *, \Rightarrow, 0, 1)$ satisfying two additional identities

$$x \wedge y = x * (x \Rightarrow y) \quad \text{(divisibility)},$$
$$(x \Rightarrow y) \vee (y \Rightarrow x) = 1 \text{ (prelinearity)}.$$

Define $\neg x = x \Rightarrow 0$. A Π-algebra (product algebra) is a BL-algebra satisfying, in addition the identities

$$x \wedge \neg x = 0,$$

$$\neg\neg x \Rightarrow (((x * z) \Rightarrow (y * z)) \Rightarrow (x \Rightarrow y)) = 1.$$

The standard Π-algebra $[0,1]_\Pi$ is the unit real interval $[0,1]$ with its usual linear order ($\wedge, \vee$ being maximum and minimum), $*$ being real product and $x \Rightarrow y = 1$ for $x \le y$, $x \Rightarrow y = y/x$ for $x > y$. $[0,1]_\Pi$ is a linearly ordered Π-algebra; each linearly ordered product algebra has Gödel negation ($\neg 0 = 1$ and $\neg x = 0$ for $x > 0$) and satisfies cancellation by a non-zero element: if $x \neq 0$ and $x * z \le y * z$ then $x \le y$.

Axioms of the product predicate logic $\Pi\forall$ are axioms of basic predicate fuzzy logic $BL\forall$ (see [1]) plus two additional axioms corresponding to the above identities, i.e.

$$(\varphi \wedge \neg\varphi) \rightarrow \bar{0},$$

$$\neg\neg\chi \rightarrow (((\varphi \& \chi) \rightarrow (\psi \& \chi)) \rightarrow (\varphi \rightarrow \psi)).$$

Deduction rules are modus ponens and generalization.

Recall the *general completeness:* $T \vdash \varphi$ over $\Pi\forall$ iff φ is true in each $\mathbf{L}$-model of T for each linearly ordered product $\mathbf{L}$.

Caution: In $\Pi\forall$, the quantifiers are not interdefinable; for a unary predicate U, the equivalence $(\forall x)U(x) \equiv \neg(\exists x)\neg U(x)$ is not provable. Moreover, there is a model $\mathbf{M}$ in which both $\neg(\forall x)U(x)$ and $\neg(\exists x)\neg U(x)$ are true (have value 1)

(U has a positive truth value for all objects, but the infimum of three values is 0). This was used in [2] to show that standard satisfiability in product logic is not arithmetical. Montagna improved my construction and showed that both satisfiability and tautologicity in both $\Pi\forall$ and $BL\forall$ is not arithmetical. We shall use Montagna's construction in the next section.

3 The Results

Theorem 1. *$\Pi\forall$ proves the formula* $(\forall x)(\varphi\&\nu) \equiv [(\forall x)\varphi\&\nu]$.

Proof: We give a semantic proof. It suffices to show for each linearly ordered Π-algebra $\mathbf{L}$, for $a_n \in L$ (n from an index set I) and for each $b \in L$ that

$$\inf_n(a_n * b) = (\inf_n a_n) * b.$$

Obviously, $\inf_n a_n * b \leq \inf_n(a_n * b)$. Conversely, let $(\inf_n a_n) * b < t$ for some $t \leq b$. Then $t = b * (b \Rightarrow t)$; write d for $b \Rightarrow t$; thus $(\inf_n a_n) * b < d * b$ and, by cancellation, $\inf_n a_n < d$, hence for some n, $a_n < d$, $a_n * b < d$, $\inf(a_n * b) < t$. For $t = \inf(a_n * b)$ we get a contradiction. □

In the rest of this section we shall construct a formula which is a standard tautology of $\Pi\forall$ but not a general tautology of $\Pi\forall$. We heavily use Montagna's construction from [4]. The reader is assumed to have [4] at his disposal.

P is a finite fragment of classical Peano arithmetic containing Robinson's Q, expressed for simplicity in the logic without function symbols (thus having a ternary predicate $A(x, y, x)$ for (the graph of) addition etc). Θ is the conjunction of the axioms of **P**. The language of **P** is called the arithmetical language. For each formula φ of the arithmetical language, φ° results from φ by replacing each atomic formula by its double negation. U is a new unary predicate; Ψ is the conjunction of Θ° with three axioms concerning U, namely $\neg(\forall x)U(x)$, $\neg(\exists x)\neg U(x)$ and an axiom expressing that with increasing x (in the arithmetical sense) the truth degree of $U(x)$ decreases quickly enough (Montagna's Φ_0, Φ_2, Φ_3; his Φ_1 is not necessary since we work with $\Pi\forall$). The following is the crucial fact about the construction:

Lemma 1. (1) For each model $\mathbf{M}$ of the extended language over the standard product algebra such that $\|\Psi\|^{\Pi}_{\mathbf{M}} > 0$, each closed formula φ of the arithmetical language satisfies

$$\|\varphi^\circ\|^{\Pi}_{\mathbf{M}} = 1 \text{ iff } \mathbf{N} \models \varphi$$

($\mathbf{N}$ is the standard crisp model of arithmetic).

(2) $\mathbf{N}$ has a $[0, 1]$-valued expansion $(\mathbf{N}, U^{\mathbf{N}})$ to a model of Ψ over the standard product algebra.

Now we apply Gödel-style diagonal lemma (cf. [3]). In **P**, arithmetize the language of $\Pi\forall$ and let $Pr_{\Pi\forall}$ be the formal provability predicate of $\Pi\forall$ expressed in **P**. Construct an arithmetical formula ν such that (**P** classically proves and hence) **N** satisfies the equivalence

$$\nu \equiv \neg Pr_{\Pi\forall}(\overline{\Psi \to \nu^\circ}).$$

(Recall that $\bar{n}$ is the n-th numeral; in **P** the above is a shorthand for the corresponding formula without function symbols.)

Lemma 2. (1) $\Pi\forall$ does not prove $\Psi \to \nu^\circ$. (2) $\mathbf{N} \models \nu$. (3) $\Psi \to \nu^\circ$ is a standard $\Pi\forall$-tautology.

Proof: (1) Assume $\Pi\forall \vdash \Psi \to \nu^\circ$; then the formulas Ψ, $\Psi \to \nu^\circ$, ν° are true in $(\mathbf{N}, U^{\mathbf{N}})$ (have the value 1), hence $\mathbf{N} \models \nu$ and thus $\Pi\forall \nvdash \Psi \to \nu^\circ$, a contradiction.

(2) Since $\Pi\forall \nvdash \Psi \to \nu^\circ$, $N \models \neg Pr_{\Pi\forall}(\overline{\Psi \to \nu^\circ})$ and thus $\mathbf{N} \models \nu$.

(3) Let $\mathbf{M}$ be a model of $\Pi\forall$ over the standard algebra. If $\|\Psi\|_{\mathbf{M}} = 0$ then $\|\Psi \to \nu^\circ\|_{\mathbf{M}} = 1$; if $\|\Psi\|_{\mathbf{M}} > 0$ then $\|\nu^\circ\|_{\mathbf{M}} = 1$ by Lemma 1 and hence $\|\Psi \to \nu^\circ\|_{\mathbf{M}} = 1$. □

Main theorem. *The formula $\Psi \to \nu^\circ$ is a standard $\Pi\forall$-tautology but not a general $\Pi\forall$-tautology.*

Proof: Immediate from the preceding lemma by the completeness theorem. □

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The Inherent Indistinguishability in Fuzzy Systems

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Abstract. This paper provides an overview of fuzzy systems from the viewpoint of similarity relations. Similarity relations turn out to be an appealing framework in which typical concepts and techniques applied in fuzzy systems and fuzzy control can be better understood and interpreted. They can also be used to describe the indistinguishability inherent in any fuzzy system that cannot be avoided.

1 Introduction

In his seminal paper on fuzzy sets L.A. Zadeh [14] proposed to model vague concepts like *big, small, young, near, far*, that are very common in natural languages, by fuzzy sets. The fundamental idea was to allow membership degrees to sets replacing the notion of crisp membership. So the starting point of fuzzy systems is the fuzzification of the mathematical concept $\in$ (*is element of*). Therefore, a fuzzy set can be seen as generalized indicator function of a set. Where a indicator function can assume only the two values zero (standing for: is not element of the set) and one (standing for: is element of the set), fuzzy sets allow arbitrary membership degrees between zero and one.

However, when we start to fuzzify the mathematical concept of being an element of a set, it seems obvious that we might also question the idea of crisp equality and generalize it to $[0,1]$-valued equalities, in order to reflect the concept of similarity. Figure 1 shows two fuzzy sets that are almost equal. From the extensional point of view, these fuzzy sets are definitely different. But from the intensional point of view in terms of modelling vague concepts they are almost equal.

In the following we will discuss the idea of introducing the concept of (intensional) fuzzified equality (or similarity). We will review some results that on the one hand show that working with this kind of similarities leads to a better

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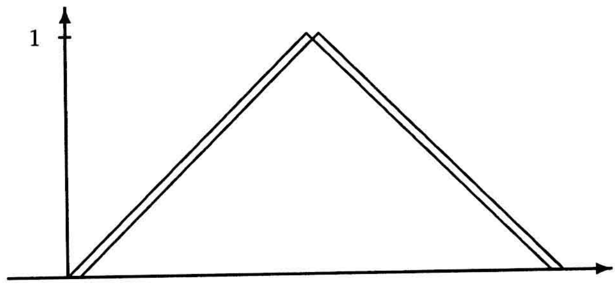

Two fuzzy sets that are almost equal.

Fig. 1. Two similar fuzzy sets

understanding of fuzzy systems and that these fuzzified equalities describe an inherent indistinguishability in fuzzy systems that cannot be overcome.

2 Fuzzy Logic

In classical logic the basics of the semantics part are truth functions for the logical connectives like $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \ldots$

Since classical logic deals with only two truth zero (false) and one (true), these truth functions can be defined in terms of simple tables as for instance for the logical connective $\wedge$ (AND):

$$\wedge : \{0,1\} \times \{0,1\} \longrightarrow \{0,1\}$$

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

In the context of fuzzy sets or fuzzy systems this restriction of a two-valued logic must be relaxed to $[0,1]$-valued logic. Therefore, the truth functions of the logical connectives must be extended from the set $\{0,1\}$ to the unit interval. Typical examples for generalized truth functions $* : [0,1] \times [0,1] \longrightarrow [0,1]$ for the logical AND $\wedge$ are:

$\alpha * \beta$	name
$\min\{\alpha, \beta\}$	minimum
$\max\{\alpha + \beta - 1, 0\}$	Łuksiewicz t-norm
$\alpha \cdot \beta$	product
$\begin{cases} \min\{\alpha,\beta\} & \text{if } \max\{\alpha,\beta\} = 1 \\ 0 & \text{otherwise} \end{cases}$	drastic product