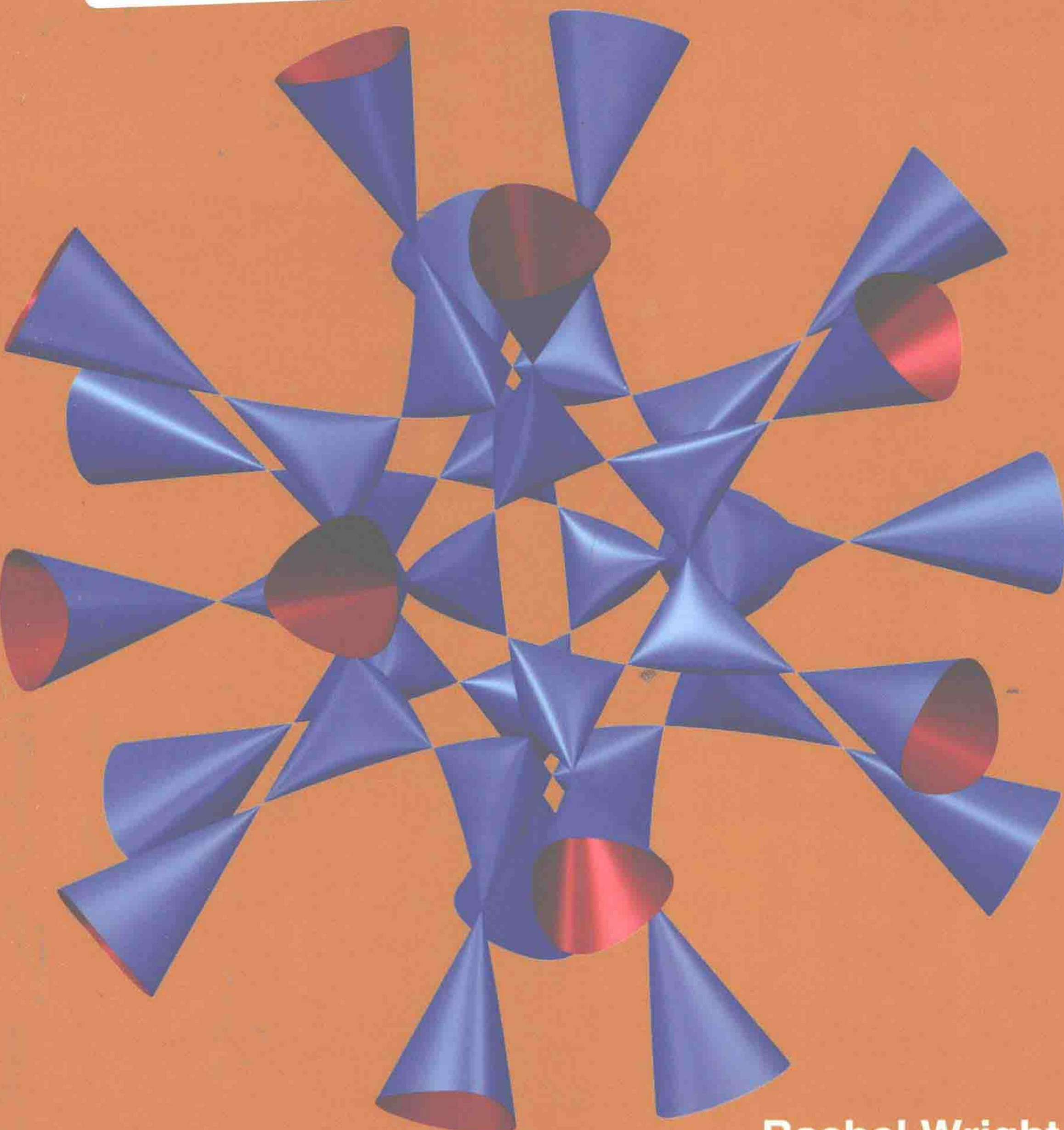
Theorems and Counterexamples in Algebraic Rings Theory



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Preface

In mathematics, and more specifically in algebra, a ring is an algebraic structure with operations generalizing the arithmetic operations of addition and multiplication. By means of this generalization, theorems from arithmetic are extended to non-numerical objects like polynomials, series, matrices and functions. Rings were first formalized as a common generalization of Dedekind domains that occur in number theory, and of polynomial rings and rings of invariants that occur in algebraic geometry and invariant theory. They are also used in other branches of mathematics such as geometry and mathematical analysis. The formal definition of rings is relatively recent, dating from the 1920s. Briefly, a ring is an abelian group with a second binary operation that is distributive over the abelian group operation and is associative. The abelian group operation is called "addition" and the second binary operation is called "multiplication" in analogy with the integers. One familiar example of a ring is the set of integers. The integers are a commutative ring, since a times b is equal to b times a. The set of polynomials also forms a commutative ring. An example of a non-commutative ring is the ring of square matrices of the same size. Finally, a field is a commutative ring in which one can divide by any nonzero element: an example is the field of real numbers.

Whether a ring is commutative or not has profound implication in the study of rings as abstract objects, the field called the ring theory. The development of the commutative theory, commonly known as commutative algebra, has been greatly influenced by problems and ideas occurring naturally in algebraic number theory and algebraic geometry: important commutative rings include fields, polynomial rings, the coordinate ring of an affine algebraic variety, and the ring of integers of a number field. On the other hand, the noncommutative theory takes examples from representation theory, functional analysis and the theory of differential operators, and the topology. In mathematics, specifically in ring theory, an algebra over a commutative ring is a generalization of the concept of an algebra over a field, where the base field K is replaced by a commutative ring R. Noetherian rings and their modules occur in many different areas of mathematics. A hundred years ago Hilbert, in the commutative setting, used properties of noetherian rings to settle a long-standing problem of invariant theory. Later, it was realised that commutative noetherian rings are one of the building blocks of modern algebraic geometry, leading to their study both abstractly and in examples. It was not until the late 1950's, with the appearance of Goldie's theorem, that it became clear that non-commutative noetherian rings constitute an interesting class of rings in their own right. As in the commutative case, non-commutative noetherian rings are studied in abstraction and in examples.

The present book explores commutative ring theory, an important a foundation for algebraic geometry and complex analytical geometry. It will be a very useful informative book for graduate students and professionals in engineering and mathematical sciences.

—*Editor*

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Chapter 1
Introduction

Ring Theory

In abstract algebra, ring theory is the study of rings—algebraic structures in which addition and multiplication are defined and have similar properties to those operations defined for the integers. Ring theory studies the structure of rings, their representations, or, in different language, modules, special classes of rings (group rings, division rings, universal enveloping algebras), as well as an array of properties that proved to be of interest both within the theory itself and for its applications, such as homological properties and polynomial identities.

Commutative rings are much better understood than noncommutative ones. Algebraic geometry and algebraic number theory, which provide many natural examples of commutative rings, have driven much of the development of commutative ring theory, which is now, under the name of *commutative algebra*, a major area of modern mathematics.

Because these three fields are so intimately connected it is usually difficult and meaningless to decide which field a particular result belongs to. For example, Hilbert's Nullstellensatz is a theorem which is fundamental for algebraic geometry, and is stated and proved in terms of commutative algebra. Similarly, Fermat's last theorem is stated in terms of elementary arithmetic, which is a part of commutative algebra, but its proof involves deep results of both algebraic number theory and algebraic geometry.

Noncommutative rings are quite different in flavour, since more unusual behaviour can arise. While the theory has developed in its own right, a fairly recent trend has sought to parallel the commutative development by building the theory of certain classes of noncommutative rings in a geometric fashion as if they were rings of functions on (non-existent) 'noncommutative spaces'.

This trend started in the 1980s with the development of noncommutative geometry and with the discovery of quantum groups. It has led to a better understanding of noncommutative rings, especially noncommutative Noetherian rings. (Goodearl 1989)

History

Commutative ring theory originated in algebraic number theory, algebraic geometry, and invariant theory. Central to the development of these subjects were the rings of integers in algebraic number fields and algebraic function fields, and the rings of polynomials in two or more variables.

Noncommutative ring theory began with attempts to extend the complex numbers to various hypercomplex number systems. The genesis of the theories of commutative and noncommutative rings dates back to the early 19th century, while their maturity was achieved only in the third decade of the 20th century.

More precisely, William Rowan Hamilton put forth the quaternions and biquaternions; James Cockle presented tessarines and coquaternions; and William Kingdon Clifford was an enthusiast of split-biquaternions, which he called *algebraic motors*. These noncommutative algebras, and the non-associative Lie algebras, were studied within universal algebra before the subject was divided into particular mathematical structure types. One sign of re-organisation was the use of direct sums to describe algebraic structure.

The various hypercomplex numbers were identified with matrix rings by Joseph Wedderburn (1908) and Emil Artin (1928). Wedderburn's structure theorems were formulated for finite-dimensional algebras over a field while Artin generalized them to Artinian rings.

In 1920, Emmy Noether, in collaboration with W. Schmeidler, published a paper about the theory of ideals in which they defined left and right ideals in a ring. The following year she published a landmark paper called *Idealtheorie in Ringbereichen*, analyzing ascending chain conditions with regard to (mathematical) ideals. Noted algebraist Irving Kaplansky called this work "revolutionary"; the publication gave rise to the term "Noetherian ring", and several other mathematical objects being called *Noetherian*.

The Development of Ring Theory

Any book on Abstract Algebra will contain the definition of a ring. It will define a ring to be a set with two operations, called addition and multiplication, satisfying a collection of axioms. These axioms require addition to satisfy the axioms for an abelian group while multiplication is associative and the two operations are connected by the distributive laws.

A ring is therefore a setting for generalising integer arithmetic. Familiar examples of rings such as the real numbers, the complex numbers, the rational numbers, the integers, the even integers, 2×2 real matrices, the integers modulo m for a fixed integer m, will almost certainly be given in the Abstract Algebra book as will many beautiful theorems on rings but what will be missing are the reasons systems satisfying these particular axioms have been singled out for such intensive study. What motivated this abstract definition of a ring?

In this article we shall be concerned with the development of the theory of commutative rings (that is rings in which multiplication is commutative) and the theory of non-commutative rings up to the 1940's. These two theories were studied quite independently of each other until about 1930 and as traces of the commutative theory appear first it is with this theory that we begin. Our comment above that study of a ring provided a generalisation of integer arithmetic is the clue to the early development of commutative ring theory. For example Legendre and Gauss investigated integer congruences in 1801. However, the motivation for generalising arithmetic came mostly from attempts-to prove Fermat's Last Theorem. This theorem, proved as recently as 1995, states:

The equation $x^n + y^n = z^n$ has no solution for positive integers x, y, z when $n > 2$.

This statement, thought to have been made in the late 1630's, was found in the marginal notes that Fermat had made in Bachet's translation of Diophantus's *Arithmetica*.

Attempts to prove this result led to proofs in the following special cases:

$n = 4$	**Fermat**	**about 1640**
$n = 3$	Euler	1753
$n = 5$	Legendre and Dirichlet	1825
$n = 14$	Dirichlet	1832
$n = 7$	Lamé	1839

Euler's work on the case $n = 3$ involved extending ordinary integer arithmetic to apply to the ring of numbers of the form $a + b\sqrt{-3}$ where a, b are integers. However, Euler failed to grasp the difficulties of working in this ring and made certain assertions which, although true, would be hard to justify.

In 1847 Lamé announced that he had a solution of Fermat's Last Theorem and sketched out a proof. Liouville suggested that the proof depended on a unique decomposition into primes which was unlikely to be true. However, Cauchy supported Lamé.

The argument which followed indicates the totally different atmosphere surrounding mathematical research of this period from that which we know today. Perhaps we could illustrate the point causing this argument. Complex numbers of the form $a + b\sqrt{-3}$, where a, b are integers, form a ring. A prime number in this ring is defined in an analogous way to a prime integer, namely a number whose only divisors of the form $a + b\sqrt{-3}$ other than itself are those numbers with multiplicative inverses. In this ring 4 can be written as a product of prime numbers in two different ways

$4 = 22$ and $4 = (1 + \sqrt{-3})\times(1 - \sqrt{-3})$.

Gauss had proved around 1801 that numbers of the form $a + b\sqrt{-1}$, where a, b are integers, could be written uniquely as a product of prime numbers of the form $a + b\sqrt{-1}$ in an analogous manner to the unique decomposition of an integer as a product of prime integers. In fact, numbers of the form $a + b + c^2$ where a, b, c are integers and is a complex cube root of 1, also have unique factorisation, and this can be used to prove the $n = 3$ case of Fermat's last Theorem.

The argument following Lamé's announcement was settled by Kummer who pointed out that he had published an example in 1844 to show that the uniqueness of such decompositions failed and in 1846 he had restored the uniqueness by introducing "ideal complex numbers". He then saw the relevance of his theory to Fermat's Last Theorem. The popular story that Kummer invented "ideal complex numbers" in an attempt to correct an error in this proof of Fermat's Last Theorem is almost certainly false. In 1847, just after Lamé's announcement, Kummer used his "ideal complex numbers" to prove Fermat's Last Theorem for all $n < 100$ except $n = 37, 59, 67$ and 74.

Up to this point we are still firmly within the realms of number theory but the genius of Dedekind pinpointed the important properties of the "ideal complex numbers". Dedekind defined an "ideal", characterising it by its now familiar properties: namely that of being a subring whose elements, on being multiplied by any ring element, remain in the subring.

Ring theory in its own right was born together with an early hint of the axiomatic method which was to dominate algebra in the 20^{th}Century. Dedekind also introduced the word "module" (early spelling: "modul") in 1871 although its initial definition was considerably more restricted than the present definition, being first introduced as a subgroup of the additive group of a ring; the term is now used for a "vector space with coefficients from a ring".

Prime numbers were generalised to prime ideals by Dedekind in 1871. A *prime ideal* is an ideal which contains the product of two elements only if it contains one of the two elements. For example all integers divisible by a fixed prime p form a prime ideal of the ring of integers. This trend towards looking at ideals rather than elements marks an important stage in the development of the theory.

In 1882 an important paper by Dedekind and Weber accomplished two things; it related geometric ideas with rings of polynomials and extended the use of modules. It is important to realise that at this stage rings of polynomials and rings of numbers were being studied, but it was to be another 40 years before an axiomatic theory of commutative rings was to be developed bringing these theories together.

Although the concept of a ring is due to Dedekind, one of the first words used was an "order" or "order-modul". This term, invented by Kronecker, is still used today in algebraic number theory. Dedekind did introduce the term "field" (*Körper*) for a commutative ring in which every non-zero element has a multiplicative inverse but the word "number ring" (*Zahlring*) or "ring" is due to Hilbert. Hilbert, motivated by studying invariant theory, studied ideals in polynomial rings proving his famous "Basis Theorem" in 1893. Special cases of this theorem had been studied by Gordan from 1868 and on seeing Hilbert's proof Gordan is said to have exclaimed "This is not mathematics, it's theology".

The decomposition of an integer into the product of powers of primes has an analogue in rings where prime integers are replaced by prime ideals but, rather surprisingly, powers of prime integers are not replaced by powers of prime ideals but rather by "primary ideals". Primary ideals were introduced in 1905 by Lasker in the context of polynomial rings. (Lasker was World Chess Champion from 1894 to

1921.) Lasker proved the existence of a decomposition of an ideal into primary ideals but the uniqueness properties of the decomposition were not proved until 1915 by Macaulay.

I D Macdonald notes in his article [2] that algebra texts such as that of Weber [4] in 1895 contained axioms for groups similar to many present-day texts. However, axioms for rings are not given by Weber and the axiomatic treatment of commutative rings was not developed until the 1920's in the work of Emmy Noether and Krull. Emmy Noether, one of the world's greatest women mathematicians, was a student of Gordan's. In about 1921 she made the important step, which we commented on earlier, of bringing the two theories of rings of polynomials and rings of numbers under a single theory of abstract commutative rings. Discrimination made it difficult for her to publish her work and it was not until Van der Waerden's important work on *Modern Algebra* [3] was published in 1930 that Noether's results become widely known.

In contrast to commutative ring theory, which as we have seen grew from number theory, non-commutative ring theory developed from an idea which, at the time of its discovery, was heralded as a great advance in applied mathematics. Hamilton attempted to generalise the complex numbers as a two dimensional algebra over the reals to a three dimensional algebra. Hamilton, who introduced the idea of a vector space, felt that this three dimensional analogue of the complex numbers would revolutionise applied mathematics but he struggled unsuccessfully with the idea for many years. In 1843 inspiration struck Hamilton - the generalisation was not to three dimensions but to four dimensions and the commutative property of multiplication no longer held. The quaternion algebra, as Hamilton called this four dimensional algebra, was widely used in applied mathematics (where it was later replaced by the vector product) and it launched non-commutative ring theory.

Matrices with their laws of addition and multiplication were introduced by Cayley in 1850 while, in 1870, Pierce noted that the now familiar ring axioms held for square matrices - another early example of the axiomatic approach to rings. The greatest early contributor to the theory of non-commutative rings was the Scottish mathematician Wedderburn. In 1905 he proved that every finite division ring (a ring in which every non-zero element has a multiplicative inverse) is commutative and so is a field. In 1908 Wedderburn had the important idea of splitting the study of a ring into two parts, one part he called the radical, the part which was left being called semi-simple. He used matrix rings to classify the semi-simple part. The importance of this work can be seen from the fact that the next 56 years were spent generalising it. We should point out that Wedderburn did not prove his results for rings but rather for hypercomplex systems - a term no longer in use which meant a finite dimensional algebra over a field.

The Wedderburn theory was extended to non-commutative rings satisfying both ascending and descending finiteness conditions (called chain conditions) by Artin in 1927. It was not until 1939 that Hopkins showed that only the descending chain condition was necessary.

Around the 1930's the theories of commutative and non-commutative rings came together and the ideas of one began to influence the other. For example, chain conditions in both commutative and non-commutative rings are investigated at much the same time. Modules, originally introduced for commutative rings, were studied for general rings. Some ideas, however, were slow to filter from one theory to the other, for example, prime ideals for non-commutative rings were not studied until 1949 by McCoy.

In the 1940's attempts were made to prove results of the Wedderburn-Artin type for rings without chain conditions. The breakthrough here was made in 1945 by Jacobson who was a student of Wedderburn's using ideas of Perlis in 1942. It is interesting to note that this fundamental work by Jacobson hinges on the idea of the "Jacobson radical" of a ring which is an analogue of a group theory idea due to Frattini as early as 1885.

Commutative Algebra

Commutative algebra is the branch of algebra that studies commutative rings, their ideals, and modules over such rings. Both algebraic geometry and algebraic number theory build on commutative algebra. Prominent examples of commutative rings include polynomial rings, rings of algebraic integers, including the ordinary integers Z, and p-adic integers.

Commutative algebra is the main technical tool in the local study of schemes.

The study of rings which are not necessarily commutative is known as noncommutative algebra; it includes ring theory, representation theory, and the theory of Banach algebras.

Overview

Commutative algebra is essentially the study of the rings occurring in algebraic number theory and algebraic geometry

In algebraic number theory, the rings of algebraic integers are Dedekind rings, which constitute therefore an important class of commutative rings. Considerations related to modular arithmetic have led to the notion of valuation ring. The restriction of algebraic field extensions to subrings has lead to the notions of integral extensions and integrally closed domains as well as the notion of ramification of an extension of valuation rings.

The notion of localization of a ring (in particular the localization with respect to a prime ideal, the localization consisting in inverting a single element and the total quotient ring) is one of the main differences between commutative algebra and the theory of non-commutative rings. It leads to an important class of commutative rings, the local rings that have only one maximal ideal. The set of the prime ideals of a commutative ring is naturally equipped with a topology, the Zariski topology. All these notions are widely used in algebraic geometry and are the basic technical tools for the definition of scheme theory, a generalization of algebraic geometry introduced by Grothendieck.

Many other notions of commutative algebra are counterparts of geometrical notions occurring in algebraic geometry. This is the case of Krull dimension, primary decomposition, regular rings, Cohen–Macaulay rings, Gorenstein rings and many other notions.

Examples

The fundamental example in commutative algebra is the ring of integers Z. The existence of primes and the unique factorization theorem laid the foundations for concepts such as Noetherian rings and the primary decomposition.

Other important examples are:

- Polynomial rings $R[x_1,...,x_n]$
- The p-adic integers
- Rings of algebraic integers.

Connections with Algebraic Geometry

Commutative algebra (in the form of polynomial rings and their quotients, used in the definition of algebraic varieties) has always been a part of algebraic geometry. However, in late 1950s, algebraic