# Calculus

Mohammed Arif

Alpha Science

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**Mohammed Arif**
Department of Mathematics
Zakir Hussain College
New Delhi



ALPHA SCIENCE INTERNATIONAL LTD.
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# Preface

This book on calculus has been written in accordance with the syllabi of B.Sc. Math honors students. The book can also be used by students with little or no background of calculus. The subject matter has been presented in a way that the students will not find any difficulty in understanding the various concepts included in the various volume. The book contains nine chapters.

The initial chapter is devoted to the various facts, especially as they appear to a beginner, of the nature of mathematics in general and of calculus in particular. The next two chapters deal systematically with the standard topics of limit and continuity and differentiability of the functions. Chapter four deals with the successive differentiation, in chapter five we have discussed the various aspects of the calculus which are generally called the backbone of the calculus. Chapter six contains on introduction of polar coordinates and conic sections, chapter seven has been devoted to the some properties of the integration. Chapter eight is hyperbolic function and last chapter nine cover the introductory knowledge of vectors.

Each chapter contains a good number of examples have taken from the question papers of different university examinations. Nearly all exercises require some thinking.

It is very much hoped that the book in its present form will help to make the study of the subject more interesting, relevant, and meaningful.

I am thankful to the publisher for their keen interest in the book.

I acknowledge with pleasure the assistance of many friends and the colleagues.

Thanks are due also to Mr. Khurram Irfan for their sincere help and interest in the computational work.

It gives me a special pleasure to express my gratitude to my wife Huma and my children Hiba and Abdul Ahad for the many ways in which they have contributed.

Suggestion for improvement will be thankfully acknowledged.

**Mohammed Arif**

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CHAPTER 1

# Preliminaries

## 1.1 REAL NUMBERS

In this chapter we present basic information that you will need for your study of calculus. We begin by discussing the real number system. The system of real numbers has evolved as a result of a process of successive extensions of the system of **natural numbers** [1, 2, 3, 4, …]. If we add two natural numbers, we get a natural number for example 6 + 2 = 8, but the inverse operation of subtraction is not always possible for example 2 – 5 is meaningless in so far as the natural numbers is concerned. Natural numbers are also referred to as positive integers. In order that the operation of subtraction performed without any restriction the natural numbers enlarge by introducing the negative integers and a number zero [0]. Thus to every positive integer [$n$] correspond a unique negative integer [$-n$] (called the additive inverse of $n$) so the relation between $n$, $-n$ and 0 as $n + (-n) = 0$, and $n + 0 = n$ for every natural number $n$. Hence the positive integers (natural numbers), the negative integers and the number zero together constitute what is known as the system of **integers** [0, ±1, ±2, ±3, ±4 …]. A **rational number** is a number that can be written as quotient of two integers, where the integer in the denominator is not zero: $r = \frac{m}{n}$ where $n \neq 0$ $\left[\frac{1}{3}, \frac{1}{2}, \frac{-3}{2}, \frac{0}{2}, 321, \ldots\right]$. Every rational number can be written as a repeating decimal for example $\frac{1}{3}$ = .33333 …, 3/11 = 0.272727 …. The rational numbers can be represented geometrically as points on a number line. The number line can be used to give us sense of order. We put a number *m* to the right of the number *n* if *m is* greater than *n*. We then write this inequality as

$$m > n$$

Similarly if *n* is greater than *m*, then *m* is to the left of *n*, and we write the inequality as

$$m < n$$

If *m* is less than or equal to *n*, that is $m < n$ or $m = n$ then we use the notation as $m \leq n$ we write $m \geq n$ to indicate that *m* is greater than or equal to *n* Hence every rational number can be represented by a point of a line, "Is the converse true?" "Is it possible to assign a rational number to every point on the number line?" The answer is no.

If we construct a square with one side of unit length, Fig. 1.1 and take a point on the number line such that *OP* is equal in length to the diagonal of this square. It will now be shown that the point *P* cannot correspond to a rational number.

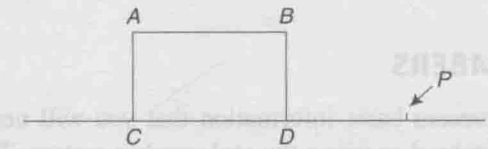


**Fig. 1.1**

Hence we see that there are so many number of points on the number line which do not correspond to any rational number. If we want to measure the length *OP* It is necessary to extend our system of numbers further by the introduction of *irrational numbers*. Thus any number that is not a rational number is called **irrational number. (the ratio between the circumference and diameter of a circle is also an irrational number)** Examples of irrational numbers are $\sqrt{2} = 1.41421356 \ldots$, $\pi = 3.141592$ (not repeated). Rational numbers and irrational numbers together constitute what is known as the system of **real numbers.**

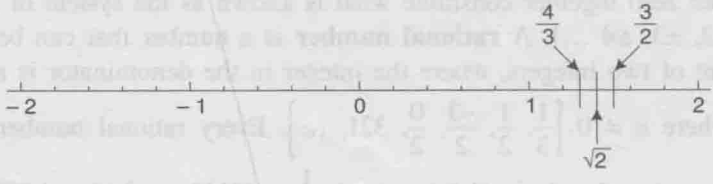


**Fig. 1.2**

| |
|---|
| Between any two real numbers, there is a rational number and an irrational number. |

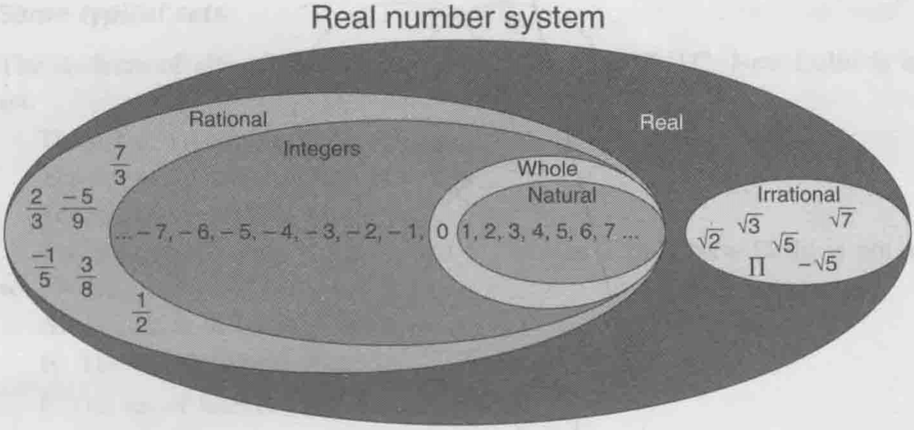


**Fig. 1.3**

## 1.2 SET

A set of objects is any well defined collection of objects, and these objects are the elements of the set. If $S$ is a set, the notation $m \in S$ means that $m$ is an element of $S$, and $m \notin S$ means that $m$ is not an element of $S$. The empty set, denoted by $\phi$, is the set containing no elements. If $S$ and $T$ are two sets then the union of $S$ and $T$, denoted by $S \cup T$ is the set of elements in $S$ or $T$ or both. That is

$$S \cup T = \{m : m \in S \text{ or } m \in T \text{ or both}\}$$

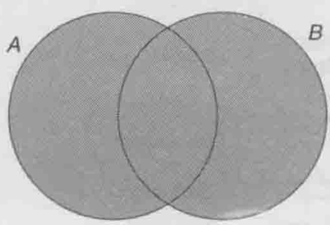


**Fig. 1.4** Shaded region is A ∪ B

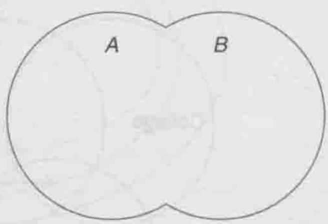


**Fig. 1.5** *A* ∪ *B*

In Boolean Logic, following UNION is represented by the intersection of two or more circles.

The intersection of $S$ and $T$ denoted by $S \cap T$ is the set of elements both in $S$ and $T$

$$S \cap T = \{m : m \in S \text{ and } m \in T\}$$

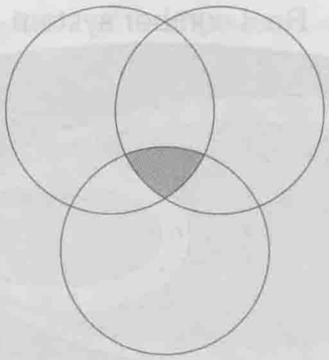

Fig. 1.6

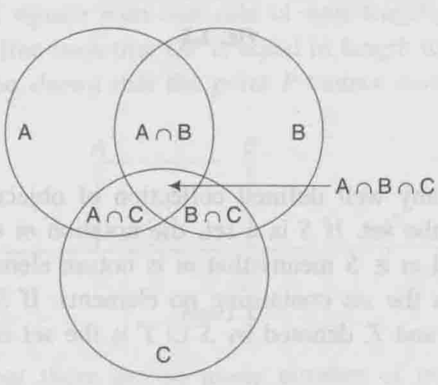


Fig. 1.7

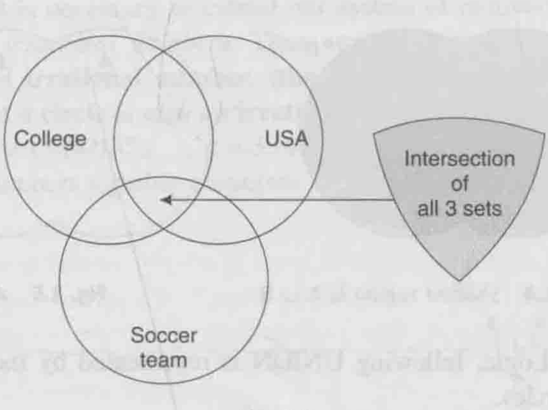


Fig. 1.8

### *Some typical sets*

The students of class B.Sc. (Hons.) Math 1st Year in Z.H.C. New Delhi is a set.

The set of all islands in Micronesia

The set of all atolls in Yap State

The set of all cars on Mokil

The students of class B.Sc. (Hons.) 1st Year in Z.H.C. New Delhi is not a set (Why).

All big cities in India is not a set (why).

N: The set of natural numbers. $\{1, 2, 3 \ldots\}$

I: The set of integers. $\{0, \pm 1, \pm 2, \pm 3 \ldots\}$

Q: The set of rational numbers. $\left\{1, 2, \frac{3}{5}, \pm\frac{7}{2}, 0 \ldots\right\}$

R: The set of real numbers. $\left\{0, 1, \pm 2, \pm 3, \sqrt{2}, \pi, \frac{3}{4} \ldots\right\}$

### *Subset*

If $S$ and $T$ are two sets such that each element of $S$ is also an element of $T$ then $S$ is called a subset of $T$ and denoted as $S \subseteq T$. i.e. the set of natural numbers is the subset of the set of integers.

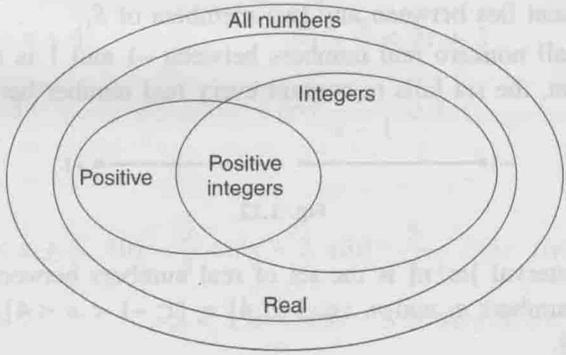


Fig. 1.9

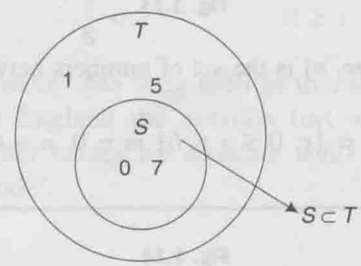


Fig. 1.10

***Equality of sets***

Two sets are said to be equal when they consist of exactly same elements. Thus, sets $S$ and $T$ are equal ($S = T$) if every element of $S$ is an element of $T$ and every element of $T$ is also an element of $S$. Thus $\{a, b, c\} = \{b, c, a\}$.

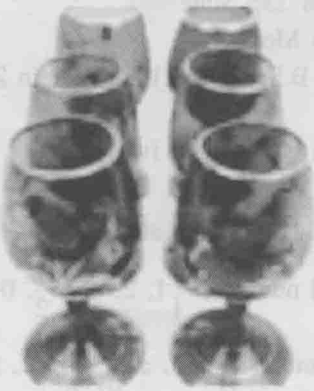

**Fig. 1.11** Six wine glasses divided into two equal sets of three

## 1.3 INTERVALS

A subset $S_1$ of $R$ is called an **interval** if $S_1$ contains at least two distinct elements and every element lies between any two members of $S_1$.

The set of all nonzero real numbers between –1 and 1 is not an interval; since 0 is absent, the set fails to contain every real number between –1 and 1. Fig. 1.12.


**Fig. 1.12**

The open interval $]m, n[$ is the set of real numbers between $m$ and $n$, not including the numbers $m$ and $n$. i.e. $]-1, 4[ = \{x: -1 < x < 4\}$ hence $m = -1$, $n = 4$. Fig. 1.13.


**Fig. 1.13**

The closed interval $[m, n]$ is the set of numbers between $m$ and $n$, including the numbers $m$ and $n$.

i.e. $[0, 6] = \{x: 0 \le x \le 6\}$ $m = 0$, $n = 6$.


**Fig. 1.14**

The half open interval $[m, n]$ is given by

$$]0, 6] = \{x{:}0 < x \leq 6\} \; m = 0, n = 6.$$


**Fig. 1.14(a)**

Interval may be infinite.

i.e. $[m, \infty[ \; = \{x{:}x \geq m\}$

i.e. $]m, \infty[ \; = \{x{:}x > m\}$

i.e. $]-\infty, m] = \{x{:}x \leq m\}$

i.e. $]-\infty, \infty[ \; = R$

The symbol and $-\infty$ denoting infinity and minus infinity, respectively are not real numbers and do not obey the usual laws of algebra, but they can be used for notational convenience.

### *Solving inequalities*

By which process we find the interval or intervals of numbers that satisfy an inequality in $x$ is called solving the inequality.

***Example 1*** Solve the following inequalities

(i) $2x - 3 < x + 4$ (ii) $-\frac{x}{5} < 3x + 2$

(iii) $\frac{5}{x+1} \geq 5$ (iv) $\frac{6}{x-1} \geq 5$

*Solution*

| | | | |
|---|---|---|---|
| (i) $2x - 3 < x + 4$ | (ii) $-\frac{x}{5} < 3x + 2$ | (iii) $\frac{5}{x+1} \geq 5$ | (iv) $\frac{6}{x-1} \geq 5$ |
| $2x < x + 7$ | $-x < 15x + 10$ | $5 \geq 5x + 5$ | $6 \geq 5x - 5$ |
| $x < 7$ | $0 < 16x + 10$ | $0 \geq 5x$ | $11 \geq 5x$ |
| | $-\frac{5}{8} < x$ | $0 \geq x$ | $\frac{11}{5} \geq x$ |

The term "absolute value" has been used in this sense since at least 1806 in French and 1857 in England the notation $|m|$ was introduced by Karl Weierstrass in 1841. Other names for *absolute value* include "the numerical value" and "the magnitude".

## 1.4 ABSOLUTE VALUE

The absolute value (or **modulus**) of a number $m$ is the distance from that number to zero and is written $|m|$. Hence 3 is 3 units from zero, so that $|3| = 3$. The number $-2$ is 2 units from zero, so that $|-2| = 2$ or $-(-2) = 2$, Fig. 1.15.


Fig. 1.15

$$|m| = m \text{ if } m \geq 0$$

$$|m| = -m \text{ if } m < 0$$

$$|m| = \begin{cases} m & m \geq 0 \\ 0 & m = 0 \\ -m & m < 0 \end{cases}$$

$$(-m)^2 = m^2 = |m|^2 \text{ or } |m| = \sqrt{m^2}$$

i.e. $$|-4| = \sqrt{(-4)^2} = 4,\ |4| = \sqrt{(4)^2} = 4,\ |-3| = \sqrt{-(3)^2} = 3$$

**Properties for absolute value:**

(i) $|-m| = m$

(ii) $|mn| = |m||n|$

(iii) $\left|\dfrac{m}{n}\right| = \dfrac{|m|}{|n|}$

(iv) $|m + n| \leq |m| + |n|$ (Triangle inequality) i.e. $|-4 + 7| = |3| < |-4| + |7| = 11$

i.e. $$|4 + 7| = |11| = |4| + |7|$$

i.e. $$|-4 - 7| = |-11| = 11 = |-4| + |-7|$$

(v) $|m - n| = 0 \Leftrightarrow m = n$

(vi) $|x| = m$ if and only if $x = \pm m$

(vii) $|x| < m$ if and only if $-m < x < m$

i.e. $$|x| < 2 \Rightarrow -2 < x < 2, \text{ Fig. 1.16}$$

Fig. 1.16

(viii) $|x| > m$ if and only if $x > m$ or $x < -m$

i.e. $$|x| > 2 \Rightarrow x > 2 \text{ or } x < -2, \text{ Fig. 1.17}$$

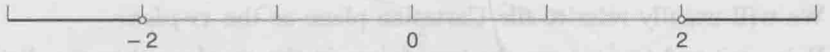


Fig. 1.17

(ix) $|x| \leq m$ if and only if $-m \leq x \leq m$

(x) $|x| \geq m$ if and only if $x \geq m$ or $x \leq -m$

(xi) $|x - m| < l \Leftrightarrow m - l < x < m + l$

***Example* 2** Solve the inequality $|x + 3| \geq 7$.

*Solution* $|x + 3| \geq 7$

$x + 3 \geq 7$ or $x + 3 \leq -7$

Hence either $x \geq 4$ or $x \leq -10$

***Example* 3** Solve the inequality $|2x - 1| \leq 3$

*Solution* $|2x - 1| \leq 3$

$-3 \leq 2x - 1 \leq 3 \Rightarrow -2 \leq 2x \leq 4 \Rightarrow -1 \leq x \leq 2.$

## 1.5 THE CARTESIAN PLANE

The Cartesian plane is named of the great French mathematician Rene Descartes. In the section 1.1 we identified the points on the line with real numbers by assigning those coordinates. Now the points in the plane can be identified with ordered pairs of real numbers. Let *OX* and *OY* be two fixed straight line perpendicular to each other. The line *OX* is called the *x-axis* while *OY* is called the *y-axis*. Both of them together are called the coordinates axes. The point *O* is termed as the origin of coordinates. Let *P* be any point in the plane, to reach this point let us draw a straight line from *P*, parallel to *OY* to meet *OX* in *M*. The distance *OM* is called *x-coordinate* (**abscissa**) and distance *MP* is called *y-coordinate* (**ordinate**) of the point *P* This ordered pair with abscissa as first member, is called the coordinate of *P* If $OM = x$, $MP = y$ then $(x, y)$ are coordinate of *P*. This coordinate system is called the rectangular coordinate system or Cartesian coordinate system. The coordinate axes of this Cartesian plane divide the plane into four regions called quadrants Fig. 1.18.

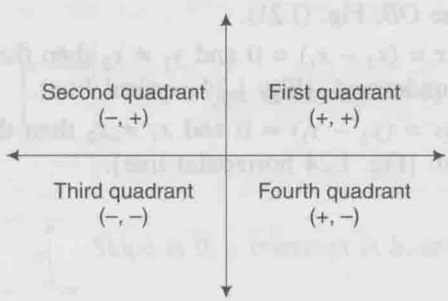


Fig. 1.18