NONLINEAR
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John H. Lowenstein

# Pseudochaotic Kicked Oscillators

## Renormalization, Symbolic Dynamics, and Transport

# 准混沌冲击振子

## 重正化、符号动力学及运动迁移现象

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Zhunhundun Chongji Zhenzi



With 86 figures, 9 of them in color

高等教育出版社·北京
HIGHER EDUCATION PRESS BEIJING

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**图书在版编目（CIP）数据**

准混沌冲击振子:重正化、符号动力学及运动迁移现象=
Pseudochaotic Kicked Oscillators:Renormalization,Symbolic Dynamics,
and Transport:英文/（美）洛文斯坦（Lowenstein, J.H.）著.
—北京:高等教育出版社,2012.3
（非线性物理科学/罗朝俊,（墨）阿弗莱诺维奇,
（瑞典）伊布拉基莫夫主编）
ISBN 978-7-04-032279-8
Ⅰ.①准… Ⅱ.①洛… Ⅲ.①动力系统（数学）-混沌
-研究-英文 Ⅳ.①O19②O415.5
中国版本图书馆 CIP 数据核字(2012)第 009873 号

策划编辑 王丽萍　　责任编辑 王丽萍　　封面设计 杨立新　　责任校对 陈旭颖
责任印制 朱学忠

| | | | |
|---|---|---|---|
| 出版发行 | 高等教育出版社 | 咨询电话 | 400-810-0598 |
| 社　　址 | 北京市西城区德外大街 4 号 | 网　　址 | http://www.hep.edu.cn |
| 邮政编码 | 100120 | | http://www.hep.com.cn |
| 印　　刷 | 涿州市星河印刷有限公司 | 网上订购 | http://www.landraco.com |
| 开　　本 | 787 × 1092　1/16 | | http://www.landraco.com.cn |
| 印　　张 | 14.25 | 版　　次 | 2012 年 3 月第 1 版 |
| 字　　数 | 230 000 | 印　　次 | 2012 年 3 月第 1 次印刷 |
| 购书热线 | 010-58581118 | 定　　价 | 59.00 元 |


物 料 号　32279-00
Sales only inside the mainland of China
{仅限中国大陆地区销售}
本书海外版由 Springer 负责在中国大陆地区以外区域销售, ISBN 为 978-3-642-28153-2

NONLINEAR PHYSICAL SCIENCE
非线性物理科学

# NONLINEAR PHYSICAL SCIENCE

*Nonlinear Physical Science* focuses on recent advances of fundamental theories and principles, analytical and symbolic approaches, as well as computational techniques in nonlinear physical science and nonlinear mathematics with engineering applications.

Topics of interest in *Nonlinear Physical Science* include but are not limited to:

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- Nonlinearity, complexity and mathematical structures in nonlinear physics
- Nonlinear phenomena and observations in nature and engineering
- Computational methods and theories in complex systems
- Lie group analysis, new theories and principles in mathematical modeling
- Stability, bifurcation, chaos and fractals in physical science and engineering
- Nonlinear chemical and biological physics
- Discontinuity, synchronization and natural complexity in the physical sciences

*Dedicated to the memory of George Zaslavsky*

# Preface

The physics of a pseudochaotic kicked oscillator could hardly be simpler: a one-dimensional harmonic oscillator, subjected to impulsive kicks in resonance with the natural frequency, with the amplitude of the kicks a periodic, piecewise linear function of the position. As with other nonlinear systems, the simple dynamics can produce, over a long time, exceedingly complex behavior, typically too complex for meaningful mathematical analysis or even reliable numerical experiments. Fortunately, a special choice of parameters provides the key to a remarkably detailed understanding of the long-time asymptotics. That key is *renormalizability*. To show how renormalizability leads to a wealth of exact results, as well as powerful computational tools for exploring long-time asymptotics, has been my principal objective in writing this monograph. The exposition is reasonably self-contained, with references to the literature where supplementary details are needed. The methods used are largely traditional ones which should be accessible to readers with a modest familiarity with the dynamics of low-dimensional Hamiltonian systems. No previous acquaintance with pseudochaos is assumed.

This book weaves together a number of important threads drawn from a long, productive, and continuing collaboration with Franco Vivaldi at Queen Mary, University of London. Our research program, initiated in 1995, has resulted in a number of published articles on the topic of piecewise isometries and their various manifestations, including Hamiltonian round-off, interval exchange transformations, kicked-oscillator models, flights, and pseudochaos. Over the years, other researchers have contributed significantly to the enterprise, notably Gullaume Poggiaspalla, Konstantin Kouptsov, Sangtian Liu, and Spyros Hatjispyros.

In writing the book, I have drawn liberally from the published results, tying them together and also adding considerable new material, and new perspectives, in order to fashion a coherent whole. Central to the story is the unifying role of the symbolic dynamics. A detailed description of the latter, including the previously unpublished derivation of admissibility rules, appears in Chapter 3. In addition, the discussion of transport in Chapter 6 is entirely new. Finally, Chapter 7 integrates the results on Hamiltonian round-off, originally published during 1997—2000, into the full story, with the inclusion of a new numerical experiment to provide fresh evidence for some of the main conclusions.

Over the years, my collaborators and I have benefited from generous short-term support of our research from the Engineering and Physical Sciences Research Council (EPSRC) and The Royal Society. The continuing hospitality and support of the Department of Physics, New York University and the School of Mathematical Sciences of Queen Mary, University of London, have been crucial to both the research program and the writing of this book.

Some of the figures in the book have been copied, with permission of the publishers, from articles in the journals *Physics Reports* (Elsevier), *Nonlinearity* (IOP), and *Communications in Nonlinear Science and Numerical Simulation* (Elsevier).

Portions of the first five chapters were originally included in a series of pedagogical lectures at the National University of Singapore (NUS) in August, 2006. I very much appreciate the hospitality and financial support provided by the Institute of Mathematical Sciences of NUS at that time.

Finally, I want to thank Valentin Afraimovich and Albert Luo for their advice and encouragement throughout the writing of this book.

New York, August, 2011 *John H. Lowenstein*

# Contents

# Chapter 1
# Introduction

In this initial chapter, we introduce a class of one-dimensional kicked oscillator models whose crystalline symmetry and renormalizability will allow us, in the course of this book, to explore the rich and varied relationships between dynamical self-similarity at the local level and transport on the infinite phase plane.

## 1.1 Kicked oscillators

The dynamics of a periodically kicked oscillator is governed by a Hamiltonian of the form (in convenient units)

$$H(x,y) = \frac{1}{2}(x^2 + y^2) - F(x) \sum_n \delta(t - 2\pi n\rho), \tag{1.1}$$

where the rotation number $\rho$ is the number of instantaneous kicks per natural period. It is assumed that the kicks are in resonance with the unperturbed oscillations, so that $\rho$ is a rational number, and that the derivative $f(x) = F'(x)$ is a periodic function of the oscillator position. Hamilton's equations of motion take the form

$$\dot{x} = \frac{\partial H}{\partial y} = y, \qquad \dot{y} = -\frac{\partial H}{\partial x} = -x + f(x) \sum_n \delta(t - 2\pi\rho n). \tag{1.2}$$

Between successive kicks, the system undergoes free oscillation, depicted in the $x, y$ phase space as uniform clockwise motion on a circular arc of arbitrary radius and angle $2\pi\rho$. This is followed by an instantaneous momentum shift $y \to y + \Delta y$, where $\Delta y$ is given by the kick function $f(x)$. In Fig. 1.1, we illustrate such a phase-space orbit for a 4-fold resonance and a sinusoidal kick function. The example is typical of kicked-oscillator models, introduced in the 1980's by Zaslavskii et al. (1986, 1991) to model the interaction of electromagnetic waves with gyrating charged particles in a plasma, and more abstractly, to illustrate the dynamical generation of crystalline and quasicrystalline order in 2-dimensional phase space.

## 1.2 Poincaré sections

The simplicity of the motion between kicks in Fig. 1.1(a) suggests that it might be advantageous to adopt a stroboscopic point of view, regarding the essence of the dynamics to be a discrete (Poincaré) map $W$ connecting the phase-space points $(x_n, y_n)$ at which the kicks are initiated. Explicitly, we have

$$W : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos 2\pi\rho & \sin 2\pi\rho \\ -\sin 2\pi\rho & \cos 2\pi\rho \end{pmatrix} \begin{pmatrix} x \\ y + f(x) \end{pmatrix}. \tag{1.3}$$

As is typical of dynamical systems with one degree of freedom and periodic forcing, the stroboscopic phase space (Poincaré section) is partitioned into disjoint invariant subsets. These subsets may be collections of points (periodic orbits), curves (quasi-periodic orbits) and more complicated sets (stochastic layers) populated in part by chaotic orbits. The discreteness of the dynamical map makes it especially easy and efficient to visualize such sets via computer-assisted iteration.

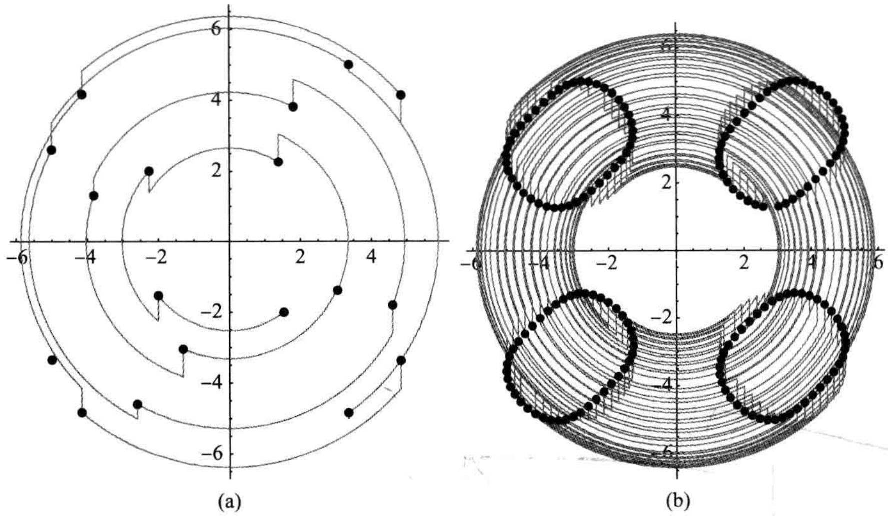


**Fig. 1.1** Quasiperiodic orbits of the sinusoidal kicked oscillator with $F(x) = -a\cos x$, $a = 0.8$, calculated over **(a)** 5 and **(b)** 50 oscillation periods. Points of the Poincaré section are shown as large dots.

Figure 1.1(b) follows the orbit of Fig. 1.1(a) through 200 kick periods (50 oscillation periods). The phase-space orbit occupies an approximately annular region with numerous self-intersections. Without emphasizing (with dots) the points of the Poincaré section, the pattern would be confusing. Further iteration would make matters worse, since we would eventually be left with a featureless annular region overlapping those of nearby orbits. The Poincaré section, on the other hand, reveals the topological simplicity of the orbit, which, viewed stroboscopically, fills out quasiperiodically four symmetrically placed closed curves. Choosing other initial conditions reveals a "phase portrait" in which the curves of the example are part of an infinite family of orbits circulating around the periodic points $(\pi,\pi)$, $(\pi,-\pi)$, $(-\pi,-\pi)$, $(-\pi,\pi)$, as shown for example, by Fig. 1.5(b) in Sect. 1.4.

For $\rho = 1/4$ in general, the Poincaré map (1.3) simplifies to

$$W\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y+f(x) \end{pmatrix} = \begin{pmatrix} y+f(x) \\ -x \end{pmatrix} \tag{1.4}$$

For reasons which will become apparent below, the map $W$ of (1.4) is often referred to the 4-*fold web map*, and we will use this terminology in the remainder of the book.

## 1.3 Crystalline symmetry

The 4-fold web map provides a simple and elegant theoretical laboratory for studying transport in a low-dimensional Hamiltonian system. This is due in large part to its *crystalline symmetry*. Specifically, suppose that the kick function $f(x)$ has period $\tau$, and that , for $\tau > 0$, $f(x)$ is continuous from the right on $[0,\tau)$, while, for $\tau < 0$, $f(x)$ is continuous from the left on $(\tau,0]$. Then every point of the real plane can be uniquely decomposed as the sum of a *local* vector $\mathbf{u}$ in the *fundamental domain*

$$\Omega = \begin{cases} [0,\tau)^2 & \tau > 0, \\ (\tau,0]^2 & \tau < 0, \end{cases}$$

and a *global* vector $\tau\mathbf{m}$, $\mathbf{m} = (m,n) \in \mathbb{Z}^2$.

From (1.4), we have, for all $\mathbf{u} \in \Omega, \mathbf{m} \in \mathbb{Z}^2$,

$$W(\mathbf{u}) + \tau\mathbf{I}\cdot\mathbf{m}, \qquad \mathbf{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.5}$$

Since $\mathbf{I}^4$ is the identity, we get from (1.5) the discrete translation invariance of the fourth iterate of $W$: for all $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{m} \in \mathbb{Z}^2$,

$$W^4(\mathbf{x} + \tau\mathbf{m}) = W^4(\mathbf{x}) + \tau\mathbf{m}. \tag{1.6}$$

Figure 1.3 illustrates the decomposition (1.5) for a hypothetical choice of $f(x)$ depicted in Fig. 1.2.

To take full advantage of the local-global decomposition, we now introduce the piecewise continuous *local map* $K : \Omega \to \Omega$, defining, for all $\mathbf{u} = (u,v) \in \Omega$,

$$K(\mathbf{u}) = W(\mathbf{u}) - \tau\mathbf{d}(\mathbf{u}), \qquad \mathbf{d}(\mathbf{u}) = (\lfloor \tau^{-1}(v + f(u)) \rfloor, -1) \in \mathbb{Z}^2,$$

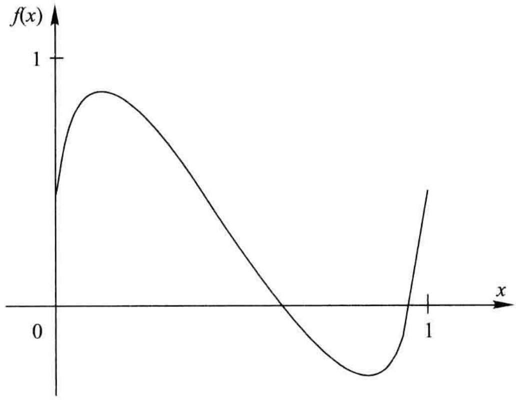


**Fig. 1.2** Graph of some kick function $f(x)$ on $[0,1)$.

so that, for all $\mathbf{u} \in \Omega$, $\mathbf{m} \in \mathbb{Z}^2$,

$$W(\mathbf{u} + \tau \mathbf{m}) = K(\mathbf{u}) + \tau \tilde{L}_{\mathbf{u}}(\mathbf{m}),$$

with the lattice isometry (rotation composed with lattice translation)

$$\tilde{L}_{\mathbf{u}}(\mathbf{m}) = \mathbf{I} \cdot \mathbf{m} + \mathbf{d}(\mathbf{u}).$$

The action of $K$ on $\Omega$, for the example of Fig. 1.3, is shown in Fig. 1.4. We note that for piecewise continuous $f(x)$, the unit square $\Omega$ is partitioned into regions $\Omega_i$, $i = 1,2,\ldots,\nu$ on which the lattice translations $\mathbf{d}(\mathbf{u})$ are constant. Thus

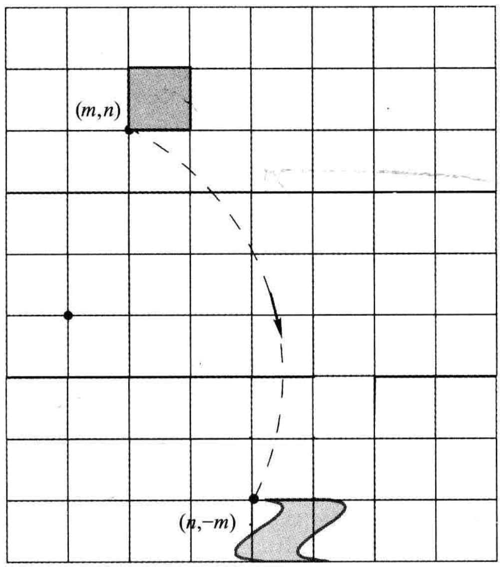


**Fig. 1.3** Action of $W(\mathbf{x})$.

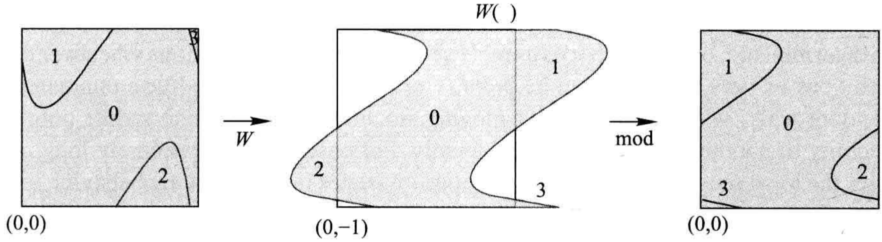


**Fig. 1.4** Construction of $K(\mathbf{u})$ for the map of Fig. 1.3.

$\tilde{L}_{\mathbf{u}}$ depends on $\mathbf{u}$ only through the index $i(\mathbf{u})$, $u \in \Omega_{i(\mathbf{u})}$, and we can express the local-global decomposition of $W(\mathbf{x})$ as

$$W(\mathbf{u}+\tau\mathbf{m}) = K(\mathbf{u}) + \tau L_{i(\mathbf{u})}(\mathbf{m}), \tag{1.7}$$

where

$$L_{i(\mathbf{u})} = \tilde{L}_{\mathbf{u}}.$$

How does the crystalline symmetry of the dynamics, expressed through (1.7), facilitate the investigation of chaotic transport? For the latter, the asymptotic long-time behavior mimics the effects of a random walk on the plane, in which successive steps are dictated by the results of independent coin tosses. Clearly it is the deterministic chaos of the local map $K$ which plays the role of the coin tosses, producing a "code" sequence $i_1, i_2, \ldots$, which in turn determines a sequence of steps on the infinite lattice. For asymptotically long times, it is the statistical distribution of the code-driven lattice coordinates which may in some sense exhibit diffusive behavior.

## 1.4 Stochastic webs

Although the main focus of this book is on maps which possess only some of the features of true chaos (hence the term pseudochaos), it is important, to properly understand the motivation for this work, that we focus first on the chaotic *stochastic web map* with kick amplitude

$$f(x) = a \sin x.$$

While the quasiperiodic orbits shown in Fig. 1.1(b) are restricted to only four cells, the same is not necessarily the case for chaotic orbits originating in the vicinity of one of the points $(m\pi, n\pi)$, $m+n$ odd. The linearized map approximating $W^4$ there is a $2 \times 2$ matrix, with one real eigenvalue greater than 1, the other less than one, so that these points are saddle points. By selecting initial points near one of these saddle points, say $(\pi, 0)$, it is easy to simulate numerically the orbits of the stochastic layer in which it is embedded. Each saddle point acts as a kind of random gate: on