AN INTRODUCTION TO SEMI-TENSOR PRODUCT OF MATRICES AND ITS APPLICATIONS

Daizhan Cheng • Hongsheng Qi • Yin Zhao

World Scientific

AN INTRODUCTION TO SEMI-TENSOR PRODUCT OF MATRICES AND ITS APPLICATIONS

Daizhan Cheng
Hongsheng Qi
Yin Zhao

Chinese Academy of Sciences, China

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

AN INTRODUCTION TO SEMI-TENSOR PRODUCT OF MATRICES AND ITS APPLICATIONS

ISBN-13 978-981-4374-68-2
ISBN-10 981-4374-68-7

Printed in Singapore by B & Jo Enterprise Pte Ltd

AN INTRODUCTION TO **SEMI-TENSOR PRODUCT OF MATRICES AND ITS APPLICATIONS**

Preface

Matrix theory has long been a fundamental tool in mathematical disciplines as well as many other scientific fields. Unlike calculus, which was mainly created by two geniuses: Isaac Newton and Gottfried Leibniz, it is hard to tell who is the principal inventor of matrix theory (or its brother — linear algebra). Though in the 19th century some mathematicians such as Carl Friedrich Gauss, Arthur Cayley, and James Joseph Sylvester *et al* have made significant contributions to it, which became the main body of matrix theory, the matrix may have appeared long before that. There is an ancient Chinese book called "The Nine Chapters on the Mathematical Art" ("Jiu Zhang Suan Shu"), that appeared in 152 BC. In this book a sequence of rectangles of data (matrices) have been used to solve linear systems. Each rectangle contains the coefficients and constants of linear systems, and the sequence of rectangles are used to describe the Gauss substitution process. In fact, a rectangle is exactly an augmented matrix. In Chinese "algebraic equation" is called "Fang Cheng", which means "rectangle process" or "rectangle transformation".

In the epoch of computer, matrix theory becomes more and more important, because it is the fundamental tool in numerical computations. Nowadays the numerical computation is not only a tool for scientific calculations, but also one of the ways to discover truths.

But the classical matrix theory also has some disadvantages.

(1) Matrix is a proper tool to deal with linear and bilinear functions. To deal with multilinear functions or even nonlinear ones matrix theory seems incapable.

(2) Comparing with scalar product, matrix product is less convenient because (i) it has dimensional restriction, and (ii) it is not commutative.

To deal with trilinear functions (or, generally, three-dimensional data), a natural way to generalize the concept of matrix is to arrange the data into a cube. Cubic matrix was firstly proposed by Bates and Watts (Bates and Watts, 1980, 1981). Then it was systemized in Tsai (1983); Wei (1986). A brief survey can be found in Wang (2002). Cubic matrix has achieved some successful applications in statistics. But since it requires several new product rules, it is not very convenient to use. As for the fourth or even higher-dimensional data, cubic matrix theory is also not applicable. To deal with general multilinear mapping, Dr. Zhang Yingshan proposed a multi-edge matrix theory (Zhang, 1993), which is a creative work. Unfortunately, because of the complexity it is not commonly accepted.

Semi-tensor product (STP) is a generalization of the conventional matrix product. It is designed to deal with higher-dimensional data as well as multilinear mappings. The basic idea of STP is motivated by computer science. In a computer the higher-dimensional data can easily be treated without arranging them into a cube or even higher-dimensional cuboid. The data are simply arranged into a long line, and the hierarchies of data are indicated by some marks. For instance, in C-language the pointer, the pointer to pointer, and the pointer to pointer to pointer, *etc.* are used to indicate the hierarchies and then to manipulate the data. The STP of matrices is designed in such a way that the product rule can automatically search the proper position for each factor of multiplier (or subset of data).

Meanwhile, unlike conventional matrix product, the STP does not require the dimension match condition. It can be used for any two matrices. (We refer to Chapter 4 for detailed discussions.) In addition, the STP has certain commutative properties, called the pseudo-commutative property, by enlarging the sizes of factor matrices and/or using an auxiliary matrix, called the swap matrix. Hence, the STP can overcome the second inadequacy of the conventional matrix product in a certain sense. Because of these advantages, the STP becomes a powerful tool in dealing with multilinear and nonlinear calculations in computers. This fact can be seen throughout this book.

Roughly speaking, this book consists of two parts. The first part introduces the concepts and properties of this generalized matrix product, and the second part is various applications, including applications to Boolean functions, cryptograph, universal algebra, physics, and algebra, *etc.* Particularly, the applications to control problems, including fuzzy control systems, control of Boolean networks, and analysis and control of nonlinear systems.

It was pointed out in Emmott (2006) that "Concepts, Theorems and Tools developed within computer science are now being developed into new conceptual tools and technological tools of potentially profound importance, with wide-ranging applications outside the subject in which they originated, especially in sciences investigating complex systems, most notably in biology and chemistry." "The invention of key new conceptual tools (*e.g.* calculus) or technological tools (*e.g.* the telescope, electron microscope) typically form the building blocks of scientific revolutions that have, historically, changed the course of history and society. Such conceptual and technological tools are now emerging at the intersection of computer science, mathematics, biology, chemistry and engineering."

We are confident that STP of matrices is one of such concepts and tools. It is motivated from computer science. Some concepts are adopted from computer science and software programming. It is applied to various problems in cooperation with numerical calculation via computer. Since the concept of STP is so natural, in applications it is so powerful, and mathematically it is so simple, we were frequently asked such questions: "Is the STP a new concept? Why it is not discovered early?" Following sceptics' clues, we have tried to find the "original version" of semi-tensor product. But we failed to find even a similar thing. It seems to us that the STP of matrices can appear only in the epoch of computer. Without computer, you can hardly find even a meaningful example for the application of STP.

Apart from other applications, the application of STP to dynamic systems is significant. It consists of two classes: (i) application to continuous dynamic systems, and (ii) application to logical dynamic systems. We refer to two books for examples of these two kinds of applications: (i) The book of Mei *et al.* (2010) shows the applications of semi-tensor product to power systems, which are typical nonlinear dynamic systems. (ii) The book of Cheng *et al.* (2011b) demonstrates the applications of semi-tensor product to the analysis and control of Boolean networks, which are typical logical dynamic systems.

A brief version of this book was prepared for a graduate course in Shandong University. Then it was expanded into a comprehensive introduction to the theory of STP and its currently known main applications. The book consists of the following contents.

Chapter 1 considers multi-dimensional data, their arrangements, and some of their operations, *etc.* First, their matrix-type arrangements are discussed. Then some nonconventional matrix products are introduced, which are Kronecker product, Hadamard product, and Khatri-Rao product. Finally, some concepts about multi-dimensional data and their properties are investigated. They are (i) tensor form; (ii) Nash equilibrium; (iii) symmetric group; (iv) swap matrix. They are fundamental tools or objects used in the sequel.

Starting from multilinear mappings, Chapter 2 proposes the left STP as a new matrix product. Its basic properties are then studied. Particularly, one sees that the STP is a generalization of the conventional matrix product and it keeps all major properties of the conventional matrix product unchanged. Then some swapping properties of STP are presented, which show that this generalization has certain pseudo-commutative properties. Finally, as a bilinear mapping, some additional properties of the STP are revealed.

Chapter 3 considers the general linear mappings between vector spaces. The STP is used as a basic tool to reconsider the cross product on $\mathbb{R}^3$, structure of general linear algebra, and the mappings over matrices. Then the conversion of different matrix expressions is considered. Finally, two applications, namely, the Lie algebras of Lie groups, and the solvability of Sylvester equations, are discussed.

In Chapter 4 we first consider an alternative STP, namely, the right STP of matrices. Its basic properties and a comparison between left and right STPs are presented. Then both left and right STPs are extended to two matrices of arbitrary dimensions.

Chapter 5 considers some further properties of the STP, which consists of rank, pseudo-inverse, and positivity of the semi-tenor product of two matrices. This chapter is based on the works of a research group in Liaocheng University.

The matrix expression of logical functions is investigated in Chapter 6, where a logical expression is converted into its matrix form (also called its algebraic form). Using its algebraic form, certain fundamental properties of logical functions are revealed. This algebraic form is then used to solve logical equations and deal with logical inferences. Finally, multi-valued logic is introduced.

Chapter 7 proposes a new type of logic, called the mix-valued logic. Its normal form is determined. Then the general logical functions and their algebraic forms are considered. Certain formulas are obtained to convert one form to the other. Finally, some applications of the mix-valued logic are briefly introduced, which include the control of fuzzy systems and the strategy description of dynamic games.

In Chapter 8 fuzzy set and fuzzy logic are investigated. First, the matrices with entries of logical variables are considered, and the operations on them are constructed. Then a finite set, its power sets, and its fuzzy sets are expressed in a uniformed vector form. Finally, the mappings over finite sets, their power sets, and their fuzzy sets are expressed into a uniformed matrix form.

Chapter 9 considers the solvability of fuzzy relational equations. Through analyzing the structure of solutions of f, a new method for obtaining parameter set solutions via STP is proposed. Then, the set of all solutions are constructed by using parameter set solutions. Numerical examples are presented to describe the method.

Chapter 10 considers the fuzzy control problem. First, the multiple fuzzy relations are introduced, the products and compounds of multiple fuzzy relations are proposed. Their matrix expressions and computations are investigated. Then they are used to multiple fuzzy inference. A new concept, called the dual fuzzy relation, is proposed to convert an infinite universe of discourse into its dual one, which is finite. Using it and the matrix expression of multiple fuzzy relations, the design of fuzzification and defuzzification for fuzzy control systems with coupled fuzzy relations is studied.

Boolean functions are particularly useful in cryptography. They are discussed in Chapter 11, in which the polynomial expression of a Boolean function is firstly considered. Based on it, the Walsh transformation and the nonlinearity of Boolean functions are investigated. The conversion back and forth between vector form and the polynomial form of Boolean functions is then investigated and formulas are obtained for numerical calculation. Finally, the results are used to investigate the symmetry of Boolean functions.

Chapter 12 considers the bi-decomposition of Boolean functions, which is particularly important in circuit design. Both disjoint and non-disjoint cases are discussed and the necessary and sufficient conditions for each cases are presented. Then the results are extended to multi-valued and mix-valued cases and the corresponding necessary and sufficient conditions are also obtained.

The Boolean calculus is discussed in Chapter 13. First, the derivative of Boolean functions is defined. Using semi-tensor product, the formulas for calculating derivatives are obtained. Then the indefinite and definite integrals of Boolean functions are proposed, and the calculating formulas are also presented. Finally, some applications, including circuit fault detection *etc.*, are discussed.

Chapter 14 introduces the applications of STP to the lattice, graph and universal algebra, and explores their relationships. First, we consider the matrix expression of lattice. Certain structure properties of lattices are investigated. Then, the graph and its adjacent matrix are analyzed. Planar graph and coloring problem are discussed. Hypergraph is also introduced. Then, the finite universal algebra is discussed. The isomorphism and homomorphism of universal algebras are investigated via structure matrices of their operators. Finally, lattice-based logics, including quantum logic, are investigated.

Chapter 15 considers the application of semi-tensor product to the analysis of Boolean networks. The algebraic form of the dynamics of Boolean networks is proposed. Using it, the topological structure of Boolean networks is investigated. "Rolling gears" structure of large Boolean networks is proposed. Finally, the normal form of dynamic-static Boolean networks is discussed by using the technique developed in Chapter 12.

Chapter 16 considers the control of Boolean networks. A framework, including state space and various subspaces, is constructed. Based on this framework, the synthesis and control of Boolean networks are studied. The basic control problems concerned in this chapter include the controllability, observability, disturbance decoupling, identification, and optimal control, *etc.*

The application of STP to game theory is discussed in Chapter 17. We consider the game with finite players and each player has only finite strategies. Assume the strategies depend on the past finite historical strategies, the strategies can be expressed as a mix-valued logical dynamic system. Then the distance in strategy space can be established. And then the Nash and sub-Nash equilibriums can be obtained by using certain strategy optimization techniques.

Chapter 18 considers the matrix expressions of multi-variable polynomials. Two basic forms are proposed, and their conversions are established. Then the differential of multi-variable smooth functions is considered. Their Taylor expressions are expressed via STP, which are exactly the same form as the one of single variable functions. The basic differential formula is obtained and then it is used to calculate Lie derivatives etc.

Chapter 19 contains some applications of STP to mathematical problems in differential geometry and algebra. The connection in a differential manifold is considered and the Christofel matrix is investigated, and its expressions under different coordinates are obtained. Then, the contraction of tensor fields is finally investigated and the formula is proved. The second part of this chapter considers the application of STP to investigating the structure and properties of finite-dimensional algebras. The structure matrix of an algebra is introduced and investigated. Then the classification and properties of two- and three-dimensional algebras are revealed. Finally, the general and product algebras are investigated. All the discussions are based on the structure matrices of algebras.

Chapter 20 considers the Morgan's problem. That is, the input-output decoupling problem for linear control systems. The problem is a long-standing open problem. A simplified equivalent formula is obtained first. Using the simplified form and the semi-tensor product, the algorithm for a numerical solution to the problem is provided.

Chapter 21 considers some linearization problems of nonlinear dynamic (control) systems. First, the Carleman's linearization is considered. Then the non-regular state feedback linearization of nonlinear control systems is investigated in detail. The problem is reduced to a single-input linearization problem. As the major result, a numerical algorithm for the non-regular state feedback linearization, is presented.

Chapter 22 considers the stabilization of nonlinear control systems. Based on the center manifold theory, the stabilization of non-minimum phase nonlinear systems is solved by designing the center manifold. Derivative homogeneous Lyapunov function is proposed and the design technique, using STP, is obtained.

Some materials of this book come from Cheng and Qi (2007) with the permission of Science Press. So, this book might be considered as a second version of Cheng and Qi (2007), though in English and completely rewritten and much enlarged and updated.

Roughly speaking, Chapters 1-4 and 18 form the fundamental theory of STP, Chapter 5 contains some additional theoretical results, which are rarely used in other chapters. And all other chapters are applications. But (1) Chapter 18 is used only for the problems with some continuous dynamics; (2) some applications are closely related. For your convenience, the relation among chapters is depicted in Fig. 0.1.

Appendix A is used to explain relevant numerical calculations. A toolbox for the algorithms can be downloaded from `http://lsc.amss.ac.cn/~dcheng/`.

The authors are indebted to Prof. L. Guo, Prof. H.-F. Chen of Chinese Academy of Sciences, and to Prof. Q. Lu of Tsinghua University for their warmhearted support. The authors would like to express their sincere thanks to their colleagues, who have made significant contributions to this new field. Some of this long list are Prof. Y. Hong of Chinese Academy of Sciences, Prof. S. Mei, Prof. F. Liu of Tsinghua University, Prof. Y. Wang,

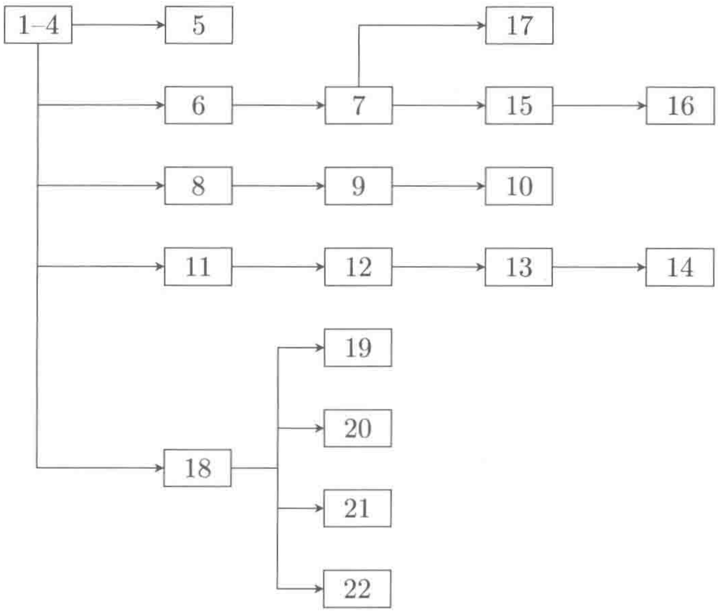

Fig. 0.1 Relation among chapters

Prof. J. Feng of Shandong University, Prof. Z. Li of Henan University of Economics and Law, Prof. J. Sun of Tongji University, Prof. J. Zhao of Liaocheng University, Prof. J. Liu of Bradley University, USA, Prof. M. Margaliot and Dr. D. Laschov of Tel Aviv University, Israel. Thanks also go to some colleagues, who have helped in preparing this manuscript. They are Dr. X. Yin of Washington University, USA, Dr. J. Kim of Glasgow University, *et al.* The works in this book are partly supported by National Natural Science Foundations of China under Grants 61074114, 60736022, 60821091 and 61104065 *etc.*

Finally, we thank World Scientific Publishing for their kind assistance in publishing this book.

Daizhan Cheng, Hongsheng Qi, and Yin Zhao
Chinese Academy of Sciences
November, 2011

Notations

$\mathbb{C}$	set of complex numbers
$\mathbb{R}$	set of real numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{Z}$	set of integers
$\mathbb{Z}_+$	set of nonnegative integers
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}_n$	finite group $\{1, \cdots, n\}$ with $+(\text{mod } n)$
$:=$	"defined as $\cdots$"
$\mathcal{M}_{m\times n}$	set of $m \times n$ real matrices
$\mathcal{M}_n$	set of $n \times n$ real matrices
$\mathbb{C}_{m\times n}$	set of $m \times n$ complex matrices
$\text{id}(i_1, \cdots, i_k; n_1, \cdots, n_k)$	ordered multi-index
$A \succ_t B$	column number of A is t times of the row number of B
$A \prec_t B$	row number of B is t times of the column number of A
$\mathcal{I}m$	set of images
δ_n^k	kth column of I_n
$\text{lcm}\{\cdot,\cdot\}$	least common multiple
$\text{gcd}\{\cdot,\cdot\}$	greatest common divisor
$\mathcal{D}$	set $\{T, F\}$ or $\{1, 0\}$
$\mathcal{D}_k$	set $\{0, \frac{1}{k-1}, \cdots, \frac{k-2}{k-1}, 1\}$
$\mathcal{D}_\infty$	set $\{r \in \mathbb{R} \| 0 \le r \le 1\}$
$\mathcal{D}_k^{m\times n}$	set $m \times n$ matrices with entries in $\mathcal{D}_k$
Δ	set $\{\delta_2^1, \delta_2^2\}$
Δ_k	set of δ_k^i, $1 \le i \le k$
$\mathcal{L}(U; V)$	set of linear mapping from U to V

$\mathcal{L}(U_1, \cdots, U_k; V)$	set of multilinear mapping from $U_1 \times \cdots \times U_k$ to V
$\mathcal{T}_t^s$	set of tensors with covariant order s and contra-variant order t
$\mathrm{Col}(A)$	set of columns of matrix A
$\mathrm{Col}_i(A)$	ith column of matrix A
$\mathrm{Row}(A)$	set of rows of matrix A
$\mathrm{Row}_i(A)$	ith row of matrix A
$\mathrm{diag}(A_1, \cdots, A_k)$	block diagonal matrix whose diagonal blocks are $A_i, i = 1, \cdots, k$
A^{-T}	$A^{-T} := (A^T)^{-1}$
A^*	$A^* := (\bar{A})^T$
$\otimes$	tensor (or Kronecker) product
$A^{\otimes k}$	$\underbrace{A \otimes \cdots \otimes A}_k$
$\ltimes$	left semi-tensor product
$\rtimes$	right semi-tensor product
$V_c(A)$	column stacking form of matrix A
$V_r(A)$	row stacking form of matrix A
$\mathrm{tr}(A)$	trace of A
$\mathbf{S}_k$	symmetric group of k elements
$W_{[m,n]}$	swap matrix with index (m, n)
$W_{[n]}$	$W_{[n]} = W_{[m,n]}$
$\mathbf{1}_k$	$\underbrace{[1, 1, \cdots, 1]}_k^T$
$m\|n$	m is a divisor of n
$\mathcal{L}_{m\times n}$	set of $m \times n$-dimensional logical matrices
$\delta_k[i_1, \cdots, i_s]$	logical matrix with $\delta_k^{i_j}$ as its jth column
$\delta_k\{i_1, \cdots, i_s\}$	$\{\delta_k^{i_1}, \cdots, \delta_k^{i_s}\} \subset \Delta_k$
$\mathcal{B}_{m\times n}$	set of $m \times n$-dimensional Boolean matrices
$\mathrm{Span} \cdots$	vector space spanned by $\cdots$
$\bowtie$	cross product on $\mathbb{R}^3$
$H < G$	H is a subgroup of G
$GL(n, \mathbb{R})$	nth order general linear group
$gl(n, \mathbb{R})$	nth order general linear algebra
$\neg$	negation
$\vee$	disjunction
$\wedge$	conjunction

$\rightarrow$	conditional
$\leftrightarrow$	biconditional
$\bar{\vee}$	exclusive or (EOR)
$\uparrow$	not and (NAND)
$\downarrow$	not or (NOR)
$\oslash$	rotator
$\triangledown$	confirmor
$\Rightarrow$	implication
$\Leftrightarrow$	equivalence
T_f	truth vector of f
$R(x_0)$	reachable set from x_0
$R_s(x_0)$	reachable set from x_0 at s steps
$\mathrm{Blk}_i(A)$	ith block of matrix A
$\mathcal{P}(E)$	set of subsets of E
$\mathcal{F}(E)$	set of fuzzy subsets of E
$\mathcal{P}(k)$	set of proper factors of $k \in \mathbb{Z}_+$
$\sqcup$	joint
$\sqcap$	meet
T_t	transient period
Ω	limit set
$(+)$	$\vee$-addition of k-valued matrices
$(\times)$	$\vee$-product of k-valued matrices
$A^{(k)}$	$\vee$-power of k-valued matrix A
$\langle + \rangle$	mod 2 addition of Boolean matrices
$\langle \times \rangle$	mod 2 product of Boolean matrices
$A^{\langle k \rangle}$	mod 2 power of Boolean matrix A
$D_v(A, B)$	vector distance of $A, B \in \mathcal{B}_{m \times n}$
$\mathcal{X}$	logical state space
$\mathcal{F}_\ell\{\cdots\}$	logical subspace generated by $\cdots$